The Seiberg-Witten Equations on Manifolds with Boundary

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Abstract

In this thesis, we undertake an in-depth study of the Seiberg-Witten equations on manifolds with boundary. We divide our study into three parts.

In Part One, we study the Seiberg-Witten equations on a compact 3-manifold with boundary. Here, we study the solution space of these equations without imposing any boundary conditions. We show that the boundary values of this solution space yield an infinite dimensional Lagrangian in the symplectic configuration space on the boundary. One of the main difficulties in this setup is that the three-dimensional Seiberg-Witten equations, being a dimensional reduction of an elliptic system, fail to be elliptic, and so there are resulting technical difficulties intertwining gauge-fixing, elliptic boundary value problems, and symplectic functional analysis.

In Part Two, we study the Seiberg-Witten equations on a 3-manifold with cylindrical ends. Here, Morse-Bott techniques adapted to the infinite-dimensional setting allow us to understand topologically the space of solutions to the Seiberg-Witten equations on a semi-infinite cylinder in terms of the finite dimensional moduli space of vortices at the limiting end. By combining this work with the work of Part One, we make progress in understanding how cobordisms between Riemann surfaces may provide Lagrangian correspondences between their respective vortex moduli spaces. Moreover, we apply our results to provide analytic groundwork for Donaldson's TQFT approach to the Seiberg-Witten invariants of closed 3-manifolds.

Finally, in Part Three, we study analytic aspects of the Seiberg-Witten equations on a cylindrical 4-manifold supplied with Lagrangian boundary conditions of the type coming from the first part of this thesis. The resulting system of equations constitute a nonlinear infinite-dimensional nonlocal boundary value problem and is highly nontrivial. We prove fundamental elliptic regularity and compactness type results for the corresponding equations, so that these results may therefore serve as foundational analysis for constructing a monopole Floer theory on 3-manifolds with boundary.

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0 Introduction

0.1 Historical Overview and Motivation

The Seiberg-Witten equations have played a fundamental role in the study of gauge theory and low-dimensional topology since their introduction in 1994 by Edward Witten. On a closed Riemannian 4-manifold, the smooth invariants one obtains from the Seiberg-Witten equations have proven to be remarkably tractable (in contrast to the invariants obtained via Donaldson theory), and by now there is a vast literature on how the Seiberg-Witten invariants behave on a wide class of 4-manifolds (see e.g. [37] for a survey). Applications of these invariants, to name a few, include a simplified proof of Donaldson's Theorem and a proof of the Thom conjecture and its generalizations. When the 4-manifold in question is a cylinder $\mathbb{R}\times Y$, one can study the Seiberg-Witten equations in the context of Floer homology and hope to produce topological invariants of the 3-manifold Y. This point of view has been brought to its full fruition through the monumental work of Kronheimer-Mrowka [21], and the study of the Seiberg-Witten-Floer invariants they define on 3-manifolds constitutes an active area of research.

Recently, there has been interest in how one may extend Floer homology theories to manifolds with boundary, particularly the Seiberg-Witten-Floer (also known as monopole Floer) homology of Kronheimer-Mrowka and also the Heegaard Floer homology of Ozsváth-Zsábo. Indeed, in the Heegaard case, there is now ample work on bordered and sutured Heegaard Floer homology, which study Heegaard Floer homology in the case when the 3-manifold has boundary, possibly equipped with some sutures on the boundary. In the monopole case, there is also a construction of sutured monopole Floer homology due to Kronheimer-Mrowka [22]. However, this latter work does not truly consist of a theory on manifolds with boundary, since the invariants one obtains consist of closing up the 3-manifold in a particular way from the sutures. In comparison, the bordered Heegaard Floer homology of Lipshitz, Ozsváth, and Thurston [26] brings in a wealth of new structures and data coming from the boundary components of the 3-manifold in question. In light of recent work by Kutluhan, Lee, and Taubes [23] in relating monopole Floer invariants and Heegaard Floer invariants on closed 3-manifolds, one might ask what the form of a "bordered monopole Floer theory" should be.

While this question is probably the most pertinent to ask given the current trends in Floer homology theory, the main inspiration for this thesis came from trying to mimic the construction of the instanton Floer homology theory of Salamon and Wehrheim [42] in the monopole Floer setting. This latter work is a Floer homology theory for the pair of a 3-manifold with boundary and a Lagrangian submanifold of the configuration space of $SU(2)$ connections on the boundary. In the instanton case, naturally occurring Lagrangians are those that are obtained by specifying a Lagrangian submanifold of the moduli space of flat $SU(2)$ connections on the boundary surface. After some considerable analytic work, Salamon and Wehrheim are able to show that the associated instanton Floer equations with Lagrangian boundary conditions yield a well-defined Floer theory. That is, one can construct a chain complex generated by gauge equivalence classes of solutions to these equations, and the homology of this complex produces an invariant of our pair of data.

One of the goals of this thesis is to explore how the above story plays out for the Seiberg-Witten equations. As we will explain in detail later on, the resulting analysis one needs to perform, both in obtaining Lagrangian data for the equations on a manifold with boundary and in studying the associated Floer boundary value problem on the cylinder, become excruciatingly more difficult in the Seiberg-Witten case. (As if the instanton Floer case were not difficult enough!) This is in sharp comparison to the case of a closed manifold, in which case the Seiberg-Witten equations are more tractable than the instanton equations due to their well-behaved compactness properties.

Nevertheless, we are able to prove a variety of results concerning the Seiberg-Witten equations on manifolds with boundary. This includes proving strong analytic results for the Seiberg-Witten equations supplied with Lagrangian boundary conditions, in particular that the resulting equations are well-posed and obey certain compactness properties. These latter results therefore constitute what is a detailed study of the analytic aspects surrounding a particularly daunting boundary value problem. We leave the potential geometric fruits of this work, which would be to complete the construction of a monopole Floer theory on 3-manifolds with boundary, to future study.

0.2 Outline of Contents and Results

We divide our study of the Seiberg-Witten equations on manifolds with boundary into four major parts:

Part I. We begin by studying the Seiberg-Witten equations on a compact 3-manifold with boundary. Here, we study these equations without imposing any boundary conditions, and thus the resulting solution spaces and their boundary values are infinite-dimensional. This makes the analysis very delicate, and because of certain analytic necessities regarding its future applications to Floer homology, we are forced to perform our analysis within a wider class of function spaces when specifying the Banach space topologies of our configurations. Specifically, because we will end up considering boundary values of Sobolev configurations with exponent $p > 2$, we are forced to work with Besov spaces, these latter spaces being boundary value spaces of Sobolev spaces. Our main result, Theorem 1.1, shows that the space of boundary values of all monopoles on a compact 3-manifold with boundary, in Besov topologies, produces for us a Lagrangian submanifold of the boundary configuration space. Some additional and rather nuanced analysis is also developed in Part I in order to understand analytic properties of these Lagrangian that are fundamental to the boundary value problem we study in Part III, which takes as input the Lagrangians obtained in Part I.

Part II. Next, we present work in progress on the Seiberg-Witten equations on a 3-manifold with cylindrical ends. Given the analysis of Part I, the main task is to understand the asymptotic behavior of finite energy solutions on the ends. In contrast to the abstract nature of Part I, which deals mainly with the abstract functional analytic nature of elliptic boundary value problems and variants of the implicit function theorem, Part II is a more concrete analysis of the Seiberg-Witten equations themselves. This is because we are required to study the Seiberg-Witten equations on a cylindrical end $[0, \infty) \times \Sigma$ as a Morse-Bott flow of a Chern-Simons-Dirac functional on the surface Σ (obtained from the Chern-Simons-Dirac functional on the 3-manifold $S^1 \times \Sigma$ by considering S^1 invariant configurations), suitably interpreted in this infinite dimensional gauge-theoretic setting. While such techniques are well documented in the instanton case (see [28], [9]), a translation of this material into the Seiberg-Witten setting appears to be absent from the literature. (In [21], only the Morse nondegenerate case is considered.) Via Theorems 7.2 and 7.6, one conclusion we arrive at is that the moduli space of finite energy monopoles on a semi-infinite cylinder $[0, \infty) \times \Sigma$ is weakly homotopy equivalent to a Hilbert bundle over the moduli space of vortices (see [14]) on the surface Σ. One should think of this moduli space as the infinite-dimensional stable manifold of the critical set of the Seiberg-Witten flow on $[0,\infty) \times \Sigma$.

At the conclusion of Part II, we piece together our analysis on a semi-infinite cylinder with the results of Part I to study the general case of a 3-manifold Y with cylindrical ends. We show that after a suitable perturbation of the Seiberg-Witten equations, the space of finite energy monopoles on Y yields an immersed Lagrangian within the product of the symplectic vortex moduli spaces associated to the ends. This result is a starting point for setting up a relationship between the Seiberg-Witten equations on 3-manifolds with boundary and Lagrangian correspondences between the vortex moduli spaces on the boundary. We also explain how our work provides analytic foundations for Donaldson's TQFT interpretation of the Seiberg-Witten invariants of a closed 3-manifold.

Part III. We now turn our study to the Seiberg-Witten equations on cylindrical 4-manifolds supplied with Lagrangian boundary conditions. What we obtain are the equations describing a Floer homology theory for the pair of a 3-manifold Y with boundary Σ and a Lagrangian submanifold $\mathfrak{L}$ of the boundary configuration space on Σ. The resulting boundary value problem is a nonlinear infinite-dimensional nonlocal boundary value problem and is therefore highly nontrivial. Nevertheless, we are able to prove the typical results that show that these equations are well-posed. Namely, we show that any weak solution to the problem is gauge equivalent to a smooth solution, that the equations obey a weak form of compactness (namely that any sequence of solutions uniformly bounded in an appropriate Sobolev norm contains a subsequence convergent modulo gauge on compact sets), and that the linearization of the equations in a suitable gauge yields a Fredholm operator. However, there is still much work to be done in order to push the work done here to obtain a Seiberg-Witten Floer theory on 3-manifolds with boundary. These issues are thoroughly discussed in Part III.

Part IV. Finally, in the last part of this thesis, we give a thorough exposition of the many tools and results from analysis that we use throughout this thesis (mainly Parts I and III). Among these tools and results are the fundamental properties of a variety of function spaces (classical function spaces, anisotropic function spaces, and vector-valued function spaces), an interpolation result for (nonlinear) Lipschitz operators between Banach spaces, and the calculus of (parameter-dependent) pseudodifferential operators in the context of elliptic boundary value problems. We use these tools to derive some analytic results which we need but which, to our knowledge, do not appear (at least explicitly) in the requisite form in the literature. The latter two topics we just described, in particular, fill in some analytic details that were omitted in [35]. Specifically, we prove Theorems 14.8 and 15.32. In hindsight, it is surprising that we had to use such a vast array of analytic tools that are not within the standard repertoire of most differential geometers, although this seems to have been a necessary consequence of the unusual nature of the difficulties involved in the Seiberg-Witten equations on manifolds with boundary (which we duly emphasize at the places where they occur). From these considerations then, Part IV is written as a purely analytic section in a sufficiently self-contained manner, both for the benefit of the reader and also as a possible reference for independent use.

The appendix to the thesis contains some additional functional analysis that we need in the thesis. These include basic properties of (symplectic) Banach spaces and their subspaces as well as fundamental relationships between Lagrangian subspaces and self-adjoint extensions of symmetric operators. The division between the material in Part IV and the appendix is not a sharp one, the only conscious difference being that the material in Part IV is developed in more technical detail and contains some results of a more specialized nature.

Note: Parts I and III are versions of the papers [34] and [35], respectively, that are adapted to this thesis.

Part I

The Seiberg-Witten Equations on a Compact 3-manifold with Boundary

1 Introduction

The Seiberg-Witten equations, introduced by Witten in [58], yield interesting topological invariants of closed three and four-dimensional manifolds and have led to many important developments in low dimensional topology during the last 15 years. On a closed 4-manifold X, the Seiberg-Witten equations are a system of nonlinear partial differential equations for a connection and spinor on X. When X is of the form $\mathbb{R} \times Y$ with Y a closed 3-manifold, a dimensional reduction leads to the 3-dimensional Seiberg-Witten equations on Y. These latter equations are referred to as the *monopole equations*. Solutions to these equations are called *monopoles*. For both three and four-dimensional manifolds, the topological invariants one obtains require an understanding of the moduli space of solutions to the Seiberg-Witten equations. On a closed 4-manifold X, the Seiberg-Witten invariant for X is computed by integrating a cohomology class over the moduli space of solutions. On a closed 3-manifold Y, the monopole invariants one obtains for Y come from studying the monopole Floer homology of Y. This involves taking the homology of a chain complex whose differential counts solutions of the Seiberg-Witten equations on $\mathbb{R} \times Y$ that connect two monopoles on Y. For further background and applications, see e.g. [21], [29], [37].

In this thesis, we study the Seiberg-Witten equations on manifolds with boundary. In Part I, we study the monopole equations on a compact 3-manifold with boundary, where no boundary conditions are specified for the equations. Specifically, as is done in the case of a closed 3-manifold, we study the geometry of the space of solutions to the monopole equations. However, unlike in the closed case, where one hopes to achieve a finite dimensional (in fact zero-dimensional) space of monopoles modulo gauge, the space of monopoles on a 3-manifold with boundary, even modulo gauge, is infinite dimensional, since no boundary conditions are imposed. Moreover, we study the space obtained by restricting the space of monopoles to the boundary. Under the appropriate assumptions (see the main theorem), we show that the space of monopoles and their boundary values are each Banach manifolds in suitable function space topologies. We should emphasize that studying the monopole equations on a 3-manifold with boundary poses some rather unusual problems. This is because the linearization of the 3-dimensional Seiberg-Witten equations are not elliptic, even modulo gauge. This is in contrast to the 4-dimensional Seiberg-Witten equations, whose moduli space of solutions on 4-manifolds with boundary has been studied in [21]. What we therefore have in our situation is a nonelliptic, nonlinear system of equations with unspecified boundary conditions. We will address the nonellipticity of these equations and other issues in the outline at the end of this introduction.

The primary motivation for studying the space of boundary values of monopoles is that the resulting space, which is a smooth Banach manifold under the appropriate hypotheses, provides natural boundary conditions for the Seiberg-Witten equations on 4-manifolds with boundary. More precisely, consider the Seiberg-Witten equations on a cylindrical 4-manifold $\mathbb{R} \times Y$, where $\partial Y = \Sigma$, and let Y' be any manifold such that $\partial Y' = -\Sigma$ and $Y' \cup_\Sigma Y$ is a smooth closed oriented Riemannian 3-manifold. We impose as boundary condition for the Seiberg-Witten equations on $\mathbb{R} \times Y$ the following: at every time $t \in \mathbb{R}$, the configuration restricted to the boundary slice $\{t\} \times \Sigma$ lies in the space of restrictions of monopoles on Y', i.e., the configuration extends to a monopole on Y'. This boundary condition has its geometric origins in the construction of a monopole Floer theory for the 3-manifold with boundary Y. We discuss these issues and the analysis behind the associated boundary value problem in Part III.

In order to state our main results, let us introduce some notation (see Section 2 for a more detailed setup). So that we may work within the framework of Banach spaces, we need to consider the completions of smooth configuration spaces in the appropriate function space topologies. The function spaces one usually considers are the standard Sobolev spaces $H^{k,p}$ of functions with k derivatives lying in L^p. However, working with these spaces alone is inadequate because the space of boundary values of a Sobolev space is not a Sobolev space (unless $p = 2$). Instead, the space of boundary values of a Sobolev space is a Besov space, and so working with these spaces will be inevitable when we consider the space of boundary values of monopoles. Thus, while we may work with Sobolev spaces on Y, we are forced to work with Besov spaces on Σ. However, to keep the analysis and notation more uniform, we will mainly work with Besov spaces on Y instead of Sobolev spaces (though nearly all of our results adapt to Sobolev spaces on Y), which we are free to do since the space of boundary values of a Besov space is again a Besov space. Moreover, since Besov spaces on 3-manifolds will be necessary for the analysis in Part III, as 3-manifolds will arise as boundaries of 4-manifolds, it is essential that we state results here for Besov spaces and not just for Sobolev spaces. On the other hand, there will be places where we want to explicitly restate[1] our results on Besov spaces in terms of Sobolev spaces (we will need both the Besov and Sobolev space versions of the analysis done in Part III), so that the separation of Besov spaces from Sobolev spaces on Y is not completely rigid, see Remark 4.17. With these considerations then, if Y is a 3-manifold with boundary Σ, consider the Besov spaces $B^{s,p}(Y)$ and $B^{s,p}(\Sigma)$, for $s \in \mathbb{R}$ and $p \geq 2$. The definition of these spaces along with their basic properties are summarized in Section 13. When $p = 2$, we have $B^{s,2} = H^{s,2}$, the usual fractional order Sobolev space of functions with s derivatives belonging to L^2 (usually denoted just H^s). For $p \neq 2$, the Besov spaces are never Sobolev spaces of functions with s derivatives in L^p. The reader unfamiliar with Besov spaces can comfortably set $p = 2$ on a first reading of Part I. Moreover, since we will be working with low fractional regularity, the

[1] Besov spaces and Sobolev spaces are "nearly identical" in the sense of Remark 13.5, so that having proven results for one of these types of spaces, one automatically obtains them for the other type.

reader may also set s equal to a sufficiently large integer to make a first reading simpler.

Let Y be endowed with a spin^c structure $\mathfrak{s}$. The spin^c structure yields for us the configuration space

$$\mathfrak{C}^{s,p}(Y) = \mathfrak{C}^{s,p}(Y, \mathfrak{s}) := B^{s,p}(\mathcal{A}(Y) \times \Gamma(\mathcal{S}))$$

on which the monopole equations are defined. Here $\Gamma(\mathcal{S})$ is the space of smooth sections of the spinor bundle $\mathcal{S} = \mathcal{S}(\mathfrak{s})$ on Y determined by $\mathfrak{s}$, $\mathcal{A}(Y) = \mathcal{A}(Y, \mathfrak{s})$ is the space of smooth spin^c connections on $\mathcal{S}$, and the prefix $B^{s,p}$ denotes that we have taken the $B^{s,p}(Y)$ completions of these spaces. The monopole equations are defined by the equations

$$SW_3(B, \Psi) = 0, \tag{1.1}$$

where SW_3 is the Seiberg-Witten map given by (2.2). Here, s and p are chosen sufficiently large so that these equations are well-defined (in the sense of distributions). Define

$$\mathfrak{M}^{s,p}(Y, \mathfrak{s}) = \{(B, \Psi) :\in \mathfrak{C}^{s,p}(Y, \mathfrak{s}) : SW_3(B, \Psi) = 0\} \tag{1.2}$$

to be space of all solutions to the monopole equations in $\mathfrak{C}^{s,p}(Y)$. Fixing a smooth reference connection $B_{\text{ref}} \in \mathcal{A}(Y)$, let

$$\mathcal{M}^{s,p}(Y, \mathfrak{s}) = \{(B, \Psi) :\in \mathfrak{C}^{s,p}(Y, \mathfrak{s}) : SW_3(B, \Psi) = 0,\ d^*(B - B_{\text{ref}}) = 0\} \tag{1.3}$$

denote the space of $B^{s,p}(Y)$ monopoles in Coulomb gauge with respect to B_{ref}.

On the boundary Σ, we can define the boundary configuration space in the $B^{s,p}(\Sigma)$ topology,

$$\mathfrak{C}^{s,p}(\Sigma) = \mathfrak{C}^{s,p}(\Sigma, \mathfrak{s}) := B^{s,p}(\mathcal{A}(\Sigma) \times \Gamma(\mathcal{S}_\Sigma)),$$

where $\mathcal{S}_\Sigma$ is the bundle $\mathcal{S}$ restricted to Σ, and $\mathcal{A}(\Sigma)$ is the space of spin^c connections on $\mathcal{S}_\Sigma$. For $s > 1/p$, we have a restriction map

$$\begin{aligned} r_\Sigma : \mathfrak{C}^{s,p}(Y) &\to \mathfrak{C}^{s-1/p,p}(\Sigma) \\ (B, \Psi) &\mapsto (B|_\Sigma, \Psi|_\Sigma) \end{aligned} \tag{1.4}$$

which restricts a connection $B \in \mathcal{A}(Y)$ and spinor $\Psi \in \Gamma(\mathcal{S})$ to Σ. Observe that when $p = 2$, this is the usual trace theorem on H^s spaces, whereby the trace of an element of $H^s(Y)$ belongs to $H^{s-1/2}(\Sigma)$. Thus, we can define the space of boundary values of the space of monopoles

$$\mathcal{L}^{s-1/p,p}(Y, \mathfrak{s}) := r_\Sigma(\mathfrak{M}^{s,p}(Y, \mathfrak{s})) \subset \mathfrak{C}^{s-1/p,p}(\Sigma). \tag{1.5}$$

We will refer to all the spaces $\mathfrak{M}^{s,p}$, $\mathcal{M}^{s,p}$, and $\mathcal{L}^{s-1/p,p}$ as *monopole spaces.*

The boundary configuration space $\mathfrak{C}(\Sigma)$ carries a natural symplectic structure. Indeed, the space $\mathfrak{C}(\Sigma)$ is an affine space modeled on $\Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma)$, and we can endow $\mathfrak{C}(\Sigma)$ with the constant symplectic form

$$\omega((a, \phi), (b, \psi)) = \int_\Sigma a \wedge b + \int_\Sigma \mathrm{Re}\,(\phi, \rho(\nu)\psi), \qquad (a, \phi), (b, \psi) \in \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma). \tag{1.6}$$

Here, $\rho(\nu)$ is Clifford multiplication by the outward unit normal ν to Σ and the inner product on spinors is induced from the Hermitian metric on $\mathcal{S}_\Sigma$. The symplectic form (1.6) extends to a symplectic form on $\mathfrak{C}^{0,2}(\Sigma)$, the L^2 configuration space on the boundary. Since $\mathfrak{C}^{s,p}(\Sigma) \subset \mathfrak{C}^{0,2}(\Sigma)$ when $s > 0$ and $p > 2$, these latter spaces are also symplectic Banach configuration spaces (in the sense of Section 19).

Let $\det(\mathfrak{s}) = \Lambda^2\mathcal{S}(\mathfrak{s})$ denote the determinant line bundle of the spinor bundle and let $c_1(\mathfrak{s}) = c_1(\det(\mathfrak{s}))$ denote its first Chern class. Then under suitable restrictions on $\mathfrak{s}$ and Y, our main theorem gives us the following relations among our Besov monopole spaces[2]:

Theorem 1.1 *(Main Theorem) Let Y be a smooth compact oriented Riemannian 3-manifold with boundary Σ and let $\mathfrak{s}$ be a spin^c structure on Y. Suppose either $c_1(\mathfrak{s})$ is non-torsion or else $H^1(Y,\Sigma) = 0$. Let $p \geq 2$ and $s > \max(3/p, 1/2)$. Then we have the following:*

(i) The spaces $\mathfrak{M}^{s,p}(Y,\mathfrak{s})$ and $\mathcal{M}^{s,p}(Y,\mathfrak{s})$ are closed[3] Banach submanifolds of $\mathfrak{C}^{s,p}(Y)$.

(ii) If furthermore, $s > 1/2 + 1/p$, then $\mathcal{L}^{s-1/p,p}(Y,\mathfrak{s})$ is a closed Lagrangian submanifold of $\mathfrak{C}^{s-1/p,p}(\Sigma)$. The restriction maps

$$r_\Sigma : \mathfrak{M}^{s,p}(Y,\mathfrak{s}) \to \mathcal{L}^{s-1/p,p}(Y,\mathfrak{s}) \tag{1.7}$$

$$r_\Sigma : \mathcal{M}^{s,p}(Y,\mathfrak{s}) \to \mathcal{L}^{s-1/p,p}(Y,\mathfrak{s}), \tag{1.8}$$

are a submersion and covering map, respectively. The fiber of (1.8) is isomorphic to the lattice $H^1(Y,\Sigma)$. In particular, if $H^1(Y,\Sigma) = 0$, then (1.8) is a diffeomorphism.

(iii) Smooth configurations are dense in $\mathfrak{M}^{s,p}(Y,\mathfrak{s})$, $\mathcal{M}^{s,p}(Y,\mathfrak{s})$, and $\mathcal{L}^{s-1/p,p}(Y,\mathfrak{s})$.

Thus, in particular, our main theorem tells us that our monopole spaces are smooth Banach manifolds for a certain range of s and p. Let us make some remarks on the condition $s > \max(3/p, 1/2)$. We need $s > 3/p$ because then $B^{s,p}(Y)$ embeds into the space $C^0(Y)$ of continuous functions on Y. This allows us to use the unique continuation results stated in Part IV. Unfortunately, for $p \leq 3$, this means we need $s > 1$, which does not seem optimal since the monopole equations only involve one derivative. For $p > 3$, we can take $s < 1$, in which case, the monopole equations are defined only in a weak sense (in the sense of distributions). We consider this low regularity case because it arises in the boundary value problem studied in Part III. Specifically, we will use the Lagrangian submanifold $\mathcal{L}^{s-1/p,p}$ as a boundary condition for the 4-dimensional Seiberg-Witten equations. Here, Lagrangian means that every tangent space to $\mathcal{L}^{s-1/p,p}$ is a Lagrangian subspace of the tangent space to

[2]The main theorem also holds with the Besov space $B^{s,p}(Y)$ on Y replaced with the function space $H^{s,p}(Y)$. The space $H^{s,p}(Y)$ is a known as a *Bessel-potential space* and for s a nonnegative integer, $H^{s,p}(Y) = W^{s,p}(Y)$ is the usual Sobolev space of functions having s derivatives in $L^p(Y)$, $1 < p < \infty$. Thus, the $H^{s,p}(Y)$ can be regarded as fractional Sobolev spaces for s not an integer. See Section 13 and Remark 4.17.

[3]For Banach submanifolds modeled on an infinite dimensional Banach space (see Definition 20.1), we use the adjective closed only to denote that the submanifold is closed as a topological subspace. For finite dimensional manifolds, closed in addition means that the manifold is compact and has no boundary. As an infinite dimensional Banach manifold is never even locally compact, this distinction in terminology should cause no confusion.

$\mathfrak{C}^{s-1/p,p}(\Sigma)$, i.e., the tangent space to $\mathcal{L}^{s-1/p,p}$ is isotropic and has an isotropic complement with respect to the symplectic form (1.6). The Lagrangian property is important because it arises in the context of self-adjoint boundary conditions. These issues will be further pursued in Part III. We should note that the analysis of the monopole equations needs to be done rather carefully at low regularity, since managing the function space arithmetic that arises from multiplying low regularity configurations becomes an important issue. In fact, the low regularity analysis is unavoidable if one wishes to prove the Lagrangian property for $\mathcal{L}^{s-1/p,p}$, since we need to understand the family of symplectic configuration spaces $\mathfrak{C}^{s-1/p,p}(\Sigma)$ as lying inside the strongly symplectic configuration space $\mathfrak{C}^{0,2}(\Sigma)$, the space of L^2 configurations on Σ (see Section 19 and also Remark 2.1). If one does not care about the Lagrangian property, then the main theorem with s large can be proven without having to deal with low regularity issues. At low regularity, the requirements $s > 1/2$ and $s > 1/2+1/p$ in the theorem are other technicalities that have to do with achieving transversality and obtaining suitable a priori estimates for monopoles (see Section 4). Let us also note that statement (iii) in the main theorem, which establishes the density of smooth monopoles in the monopole spaces, is not at all obvious. Indeed, our monopole spaces are not defined to be Besov closures of smooth monopoles, but as seen in (1.2), they arise from the zero set of the map SW_3 defined on a Banach space of configurations. This way of defining our monopole spaces is absolutely necessary if we are to use the essential techniques from Banach space theory, such as the inverse function theorem. However, since our monopole spaces are not linear Banach spaces, and since they are infinite dimensional modulo gauge, some work must be done to show that a Besov monopole can be approximated by a smooth monopole.

Let us make the simple remark that our theorem is nonvacuous due to the following example:

Example. Suppose $c_1(\mathfrak{s})$ is torsion. Then every flat connection on $\det(\mathfrak{s})$ yields a solution of the monopole equations (where the spinor component is identically zero). If $H^1(Y,\Sigma) = 0$, the main theorem implies that the monopole spaces are smooth nonempty Banach manifolds. In fact, using Theorem 4.8, one can describe a neighborhood of any configuration in the space of monopoles on Y, in particular, a neighborhood of a flat connection.

Our main theorem will be proven in Theorems 4.2 and 4.13. In addition to these, we have Theorems 4.8 and 4.15, which describe for us certain smoothing properties of the local chart maps of our monopole spaces. These properties are not only of interest in their own right, since our monopole spaces are infinite dimensional Banach manifolds, but they will play an essential role in Part III.

Finally, let us also remark that our methods, and hence our theorems, carry over straightforwardly if we perturb the Seiberg-Witten equations by a smooth coclosed 1-form η. That is, we consider the equations

$$SW_3(B,\Psi) = (\eta, 0). \tag{1.9}$$

We have the following result:

Corollary 1.2 *Suppose either* $c_1(\mathfrak{s}) \neq 2[*\eta]$ *or else* $H^1(Y,\Sigma) = 0$. *Then all the conclu-*

sions of the main theorem remain true for the monopole spaces associated to the perturbed monopole equations (1.9).

Thus, for any $\mathfrak{s}$, the corresponding perturbed spaces of monopoles will be smooth for generic coclosed perturbations. Moreover, these monopole spaces will be nonempty for many choices of η, since given any smooth configuration (B, Ψ) such that Ψ lies in the kernel of D_B, the Dirac operator determined by B (see Section 2), we can simply define η to be the value of $SW_3(B, \Psi)$, in which case (B, Ψ) automatically solves (1.9).

Outline: Part I is organized as follows. In Section 2, we define the basic setup for the monopole equations on Y. In Section 3, we establish the foundational analysis to handle the linearization of the monopole equations. This primarily involves understanding the various gauge fixing issues involved as well as understanding how elliptic operators behave on manifolds with boundary. The presence of a boundary makes this latter issue much more difficult than the case when there is no boundary. Indeed, on a closed manifold, elliptic operators are automatically Fredholm when acting between standard function spaces (e.g. Sobolev spaces and Besov spaces). On the other hand, on a manifold with boundary, the kernel of an elliptic operator is always infinite dimensional. To fully understand the situation, we need to use the pseudodifferential tools summarized in Section 15, which allows us to handle elliptic boundary value problems on a variety of function spaces, in particular, Besov spaces of low regularity. From this, what we will find is that the tangent spaces to our monopole spaces are given essentially by the range of pseudodifferential projections. Having established the linear theory, we use it in Section 4 to study the nonlinear monopole equations and prove our main results concerning the monopole spaces.

As we we pointed out earlier, the linearization of the 3-dimensional Seiberg-Witten equations are unfortunately not elliptic, even modulo gauge. To work around this, we embed these equations into an elliptic system and use tools from elliptic theory to derive results for the original equations from the enlarged system. This procedure is described in Section 3.3, where issues regarding ellipticity and gauge-fixing intertwine. Furthermore, when we restrict to the boundary, passing from the enlarged elliptic system back to the original non-elliptic system involves a symplectic reduction, and so there is also an important interplay of symplectic functional analysis in what we do.

2 The Basic Setup

We give a quick overview of the setup for the Seiberg-Witten equations on a 3-manifold. For a more detailed setup, see [21]. Let Y be a smooth compact oriented Riemannian 3-manifold with boundary Σ. A spin^c structure $\mathfrak{s}$ on Y is a choice of $U(2)$ principal bundle over Y that lifts the $SO(3)$ frame bundle of Y. The space of all spin^c structures on Y is a torsor over $H^1(Y;\mathbb{Z})$. Any given spin^c structure $\mathfrak{s}$ determines for us a spinor bundle $\mathcal{S} = \mathcal{S}(\mathfrak{s})$ over Y, which is the two-dimensional complex vector bundle over Y associated to the $U(2)$ bundle corresponding to $\mathfrak{s}$. Endow $\mathcal{S}$ with a Hermitian metric. From this, we obtain Clifford multiplication bundle maps $\rho : TY \to \mathrm{End}(\mathcal{S})$ and $\rho : T^*Y \to \mathrm{End}(\mathcal{S})$, where the two are intertwined by the fact that the Riemannian metric gives a canonical isomorphism $TY \cong T^*Y$. The map ρ extends complex linearly to a map on the complexified exterior algebra of T^*Y and we choose ρ so that ρ maps the volume form on Y to the identity automorphism on $\mathcal{S}$. This determines the spinor bundle $\mathcal{S} = (\mathcal{S}, \rho)$ uniquely up to isomorphism.

Fix a spin^c structure $\mathfrak{s}$ for the time being on Y. Only later in Section 4 will be impose restrictions on $\mathfrak{s}$. A spin^c connection on $\mathcal{S}$ is a Hermitian connection ∇ on $\mathcal{S}$ for which Clifford multiplication is parallel, i.e., for all $\Psi \in \Gamma(\mathcal{S})$ and $e \in \Gamma(TY)$, we have $\nabla(\rho(e)\Psi) = \rho(\nabla^{LC}e)\Psi + \rho(e)\nabla\Psi$, where ∇^{LC} denotes the Levi-Civita connection. Let $\mathcal{A}(Y)$ denote the space of spin^c connections $\mathcal{A}(Y)$ on Y. The difference of any two spin^c connections acts on a spinor via Clifford multiplication by an imaginary-valued 1-form. Thus, given any fixed spin^c connection $B_0 \in \mathcal{A}(Y)$, we can identify

$$\mathcal{A}(Y) = \{B_0 + b : b \in \Omega^1(Y; i\mathbb{R})\},$$

so that $\mathcal{A}(Y)$ is an affine space over $\Omega^1(Y; i\mathbb{R})$.

Let

$$\mathfrak{C}(Y) = \mathfrak{C}(Y, \mathfrak{s}) = \mathcal{A}(Y) \times \Gamma(\mathcal{S})$$

denote the configuration space of all smooth spin^c connections and smooth sections of the spinor bundle $\mathcal{S}$. It is an affine space modeled on $\Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S})$. By abuse of notation we let the inner product $(\cdot, \cdot)$ denote the following items: the Hermitian inner product on $\mathcal{S}$, linear in the first factor, the Hermitian inner product on complex differential forms induced from the Riemannian metric on Y, and finally the real inner product on $\Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S})$ induced from the real part of the inner products on each factor.

The Seiberg-Witten equations on Y are given by the pair of equations

$$\begin{aligned} \frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0 &= 0 \\ D_B\Psi &= 0, \end{aligned} \tag{2.1}$$

where $(B, \Psi) \in \mathfrak{C}(Y)$. Here B^t is the connection induced from B on the determinant line bundle $\det(\mathfrak{s}) = \Lambda^2(\mathcal{S})$ of $\mathcal{S}$, the element $F_{B^t} \in \Omega^2(Y, i\mathbb{R})$ is its curvature, and $*$ is the Hodge star operator on Y. For any spinor Ψ, the term $(\Psi\Psi^*)_0 \in \mathrm{End}(\mathcal{S})$ is the trace-free Hermitian endomorphism of $\mathcal{S}$ given by the trace-free part of the map $\varphi \mapsto (\varphi, \Psi)\Psi$. Since ρ maps $\Omega^1(Y; i\mathbb{R})$ isomorphically onto the space of trace-free Hermitian endomorphisms of $\mathcal{S}$, then $\rho^{-1}(\Psi\Psi^*)_0 \in \Omega^1(Y; i\mathbb{R})$ is well-defined. Finally, $D_B : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ is the spin^c Dirac operator associated to the spin^c connection B, i.e., in local coordinates, we have $D_B = \sum_{i=1}^3 \rho(e_i)\nabla_{B,e_i}$ where ∇_B is the spin^c covariant derivative associated to B and the e_i form a local orthonormal frame of tangent vectors.

Altogether, the left-hand side of (2.1) defines for us a Seiberg-Witten map

$$\begin{aligned} SW_3 : \mathfrak{C}(Y) &\to \Omega^1(Y; i\mathbb{R}) \times \Gamma(\mathcal{S}) \\ (B, \Psi) &\mapsto \left(\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0, D_B\Psi\right). \end{aligned} \tag{2.2}$$

Thus, solutions to the Seiberg-Witten equations are precisely the zero set of the map SW_3.

We will refer to a solution of the Seiberg-Witten equations as a *monopole.* Let

$$\mathfrak{M}(Y, \mathfrak{s}) = \{(B, \Psi) \in \mathfrak{C}(Y) : SW_3(B, \Psi) = 0\} \tag{2.3}$$

denote the solution space of all monopoles on Y. Fixing a smooth reference connection $B_{\mathrm{ref}} \in \mathcal{A}(Y)$ once and for all, let

$$\mathcal{M}(Y, \mathfrak{s}) = \{(B, \Psi) \in \mathfrak{C}(Y) : SW_3(B, \Psi) = 0, d^*(B - B_{\mathrm{ref}}) = 0\} \tag{2.4}$$

denote the space of all smooth monopoles that are in Coulomb gauge with respect to B_{ref}. Without any assumptions, the spaces $\mathfrak{M}(Y, \mathfrak{s})$ and $\mathcal{M}(Y, \mathfrak{s})$ are just sets, but we will see later, by transversality arguments, that these spaces of monopoles are indeed manifolds under suitable assumptions on Y and $\mathfrak{s}$. Since $\partial Y = \Sigma$ is nonempty and no boundary conditions have been specified for the equations defining $\mathfrak{M}(Y, \mathfrak{s})$ and $\mathcal{M}(Y, \mathfrak{s})$, these spaces will be infinite dimensional, even modulo the full gauge group. Note that the space $\mathcal{M}(Y, \mathfrak{s})$ is obtained from $\mathfrak{M}(Y, \mathfrak{s})$ through a partial gauge-fixing, see Section 3.1.

Let the boundary Σ be given the usual orientation induced from that of Y, i.e., if ν is the outward normal vector field along Σ and dV is the oriented volume form on Y, then $\nu \lrcorner dV$ yields the oriented volume form on Σ. On the boundary Σ, we have the configuration space

$$\mathfrak{C}(\Sigma) = \mathcal{A}(\Sigma) \times \Gamma(\mathcal{S}_\Sigma),$$

where S_Σ is the bundle $\mathcal{S}$ restricted to Σ, and $\mathcal{A}(\Sigma)$ is the space of spin^c connections on S_Σ. We have a restriction map

$$\begin{aligned} r_\Sigma : \mathfrak{C}(Y) &\to \mathfrak{C}(\Sigma) \\ (B, \Psi) &\mapsto (B|_\Sigma, \Psi|_\Sigma) \end{aligned} \tag{2.5}$$

From this, we can define the space of (tangential) boundary values of the space of monopoles

$$\mathcal{L}(Y, \mathfrak{s}) = r_\Sigma(\mathfrak{M}(Y, \mathfrak{s})). \tag{2.6}$$

Observe that the space $\mathcal{L}(Y, \mathfrak{s})$ is nonlocal in the sense that its elements, which belong to $\mathfrak{C}(\Sigma)$, are not defined by equations on Σ. Indeed, $\mathcal{L}(Y, \mathfrak{s})$ is determined by the full Seiberg-Witten equations in the interior of the manifold. This makes the analysis concerning the manifold $\mathcal{L}(Y, \mathfrak{s})$ rather delicate, since one has to control both the space $\mathfrak{M}(Y, \mathfrak{s})$ and the behavior of the map r_Σ.

Ultimately, we want our manifolds to be Banach manifolds, and so we must complete our smooth configuration spaces in the appropriate function space topologies. As explained in the introduction, the topologies most suitable for us are the Besov spaces $B^{s,p}(Y)$ and $B^{s,p}(\Sigma)$ on Y and Σ, respectively, where $s \in \mathbb{R}$ and $p \geq 2$. These are the familiar H^s spaces when $p = 2$ and for $p \neq 2$, the Besov spaces are never Sobolev spaces, i.e., spaces of functions with a specified number of derivatives lying in L^p. Nevertheless, much of the analysis we will do applies to Sobolev spaces as well, since the analysis of elliptic boundary value problems is flexible and applies to a wide variety of function spaces. To keep the notation minimal, we work mainly with Besov spaces and make a general remark at the end about how statements generalize to Sobolev spaces and other spaces (see Remark 4.17).

The Besov spaces, other relevant function spaces, and their properties are summarized in Part IV. On a first reading, one may set $p = 2$ and s a large number, say a large integer, wherever applicable, so that the function spaces are as familiar as desired.

Thus, for $p \geq 2$ and $s \in \mathbb{R}$, we consider the Besov spaces $B^{s,p}(Y)$ and $B^{s,p}(\Sigma)$ of scalar-valued functions on Y and Σ, respectively. These topologies induce topologies on vector bundles over Y and Σ in the natural way, and so we may define the Besov completions of the configuration spaces

$$\mathfrak{C}^{s,p}(Y) = B^{s,p}(Y) \text{ closure of } \mathfrak{C}(Y) \tag{2.7}$$
$$\mathfrak{C}^{s,p}(\Sigma) = B^{s,p}(\Sigma) \text{ closure of } \mathfrak{C}(\Sigma). \tag{2.8}$$

Of course, when defining Besov norms on the space of connections in the above, we have to first choose a (smooth) reference connection, which then identifies the Besov space of connections with the Besov space of 1-forms.

For s, p such that the Seiberg-Witten equations make sense on $\mathfrak{C}^{s,p}(Y)$ (in the sense of distributions), we have the monopole spaces

$$\mathfrak{M}^{s,p}(Y, \mathfrak{s}) = \{(B, \Psi) \in \mathfrak{C}^{s,p}(Y) : SW_3(B, \Psi) = 0\} \tag{2.9}$$
$$\mathcal{M}^{s,p}(Y, \mathfrak{s}) = \{(B, \Psi) \in \mathfrak{C}^{s,p}(Y) : SW_3(B, \Psi) = 0, d^*(B - B_{\mathrm{ref}}) = 0.\} \tag{2.10}$$

in $\mathfrak{C}^{s,p}(Y)$. Observe that for the range of s and p that are relevant for us, namely $p \geq 2$ and $s > \max(3/p, 1/2)$, the Seiberg-Witten equations are well-defined on $\mathfrak{C}^{s,p}(Y)$. This follows from Corollary 13.14 and Theorem 13.18.

For $s > 1/p$, the restriction map (2.5) extends to a map

$$r_\Sigma : \mathfrak{C}^{s,p}(Y) \to \mathfrak{C}^{s-1/p,p}(\Sigma), \tag{2.11}$$

and so we can define

$$\mathcal{L}^{s-1/p,p}(Y, \mathfrak{s}) := r_\Sigma(\mathfrak{M}^{s,p}(Y, \mathfrak{s})).$$

Having defined our monopole spaces in the relevant topologies, we now begin the study of their properties as Banach manifolds. With $\mathfrak{s}$ and Y fixed, we will often write $\mathfrak{M}(Y)$ or simply $\mathfrak{M}$ instead of $\mathfrak{M}(Y, \mathfrak{s})$. Likewise for the other monopole spaces.

Remark 2.1 In the remainder of Part I, we will be stating results for various values of s and p. Unless stated otherwise, we will always assume

$$2 \leq p < \infty. \tag{2.12}$$

Many of the statements of Part I are phrased in such a way that the range of permissible s and p is quite large and moreover, several topologies are often simultaneously involved (e.g. Lemma 3.4). This is not merely an exercise in function space arithmetic and there are several important reasons for stating our results this generally.

First, we will need to work in the low regularity regime with $s < 1$ for applications in Part III. In particular, when a first order operator acts on a configuration with regularity $s < 1$, we obtain a configuration with negative regularity and hence our results must be stated in enough generality to account for this. Second, as mentioned in the introduction, the

Lagrangian property of $\mathcal{L}^{s-1/p,p}$, even at high regularity (i.e. large s), requires an analysis of the $\mathcal{L}^{s-1/p,p}$ at low regularity. Indeed, among all the spaces $\mathfrak{C}^{s,p}(\Sigma)$, only $\mathfrak{C}^{0,2}(\Sigma)$ is modeled on a strongly symplectic Hilbert space (see Section 19), and we will need to study all the symplectic spaces $\mathfrak{C}^{s,p}(\Sigma)$, $s > 0$, as subspaces of the space $\mathfrak{C}^{0,2}(\Sigma)$. Thus, in a fundamental way, we will generally be considering multiple topologies simultaneously. Observe that from these considerations, it is necessary to have the pseudodifferential tools summarized in Section 15. Indeed, we need to understand elliptic boundary value problems at low (even negative) regularity, and furthermore, we have to deal with the fact that there is no trace map $\mathfrak{C}^{1/2,2}(Y) \to \mathfrak{C}^{0,2}(\Sigma)$.

Hence, it is natural to state our results for a range of s and p that are as flexible as possible. In fact, based on the function space arithmetic alone, many of the proofs involved are natural for the range $s > 3/p$ say (since then $B^{s,p}(Y)$ is an algebra), and it would be unnatural to restrict the range of s based on the particular applications we have in mind. Finally, it may be desirable to sharpen the range of s and p considered in Part I and so we try to state our results in a sufficiently general way at the outset.

Notation. Given any space $\mathfrak{X}$ of configurations over a manifold $X = Y$ or Σ, we write $B^{s,p}\mathfrak{X}$ to denote the closure of $\mathfrak{X}$ with respect to the $B^{s,p}(X)$ topology. We define $L^p\mathfrak{X}$, $C^0\mathfrak{X}$, and $H^{s,p}\mathfrak{X}$ similarly. For brevity, we may refer to just the function space which defines the topology of a configuration, e.g., if $\mathfrak{X}$ is a space of configurations on Y, we may say an element $u \in B^{s,p}\mathfrak{X}$ belongs to $B^{s,p}(Y)$ or just $B^{s,p}$ for short. If E is a vector bundle over a space X, we write $B^{s,p}(E)$ as shorthand for $B^{s,p}\Gamma(E)$, the closure of the space $\Gamma(E)$ of smooth sections of E in the topology $B^{s,p}(X)$. If X has boundary, we write $E_{\partial X}$ to denote $E|_{\partial X}$, the restriction of the bundle E to the boundary ∂X.

From now on, we will make free use of the basic properties of the function spaces employed in Part I (multiplication and embedding theorems in particular), all of which can be found in the Part IV.

3 Linear Theory

To study our monopole spaces, we first study their linearization, that is, their formal tangent spaces. This involves studying the linearization of the Seiberg-Witten map. Furthermore, since we have an action of a gauge group, we must take account of this action in our framework. This section therefore splits into three subsections. In the first section, we study the gauge group and how it acts on the space of configurations. Next, we study how this action decomposes the tangent space to the configuration space into natural subspaces. Finally, we apply these decompositions to the study of the linearized Seiberg-Witten equations, where modulo gauge and other modifications, we can place ourselves in an elliptic situation.

3.1 The Gauge Group

The gauge group $\mathcal{G} = \mathcal{G}(Y) = \mathrm{Maps}(Y, S^1)$ is the space of smooth maps $g : Y \to S^1$, where we regard $S^1 = \{e^{i\theta} \in \mathbb{C} : 0 \leq \theta < 2\pi\}$. Elements of the gauge group act on $\mathfrak{C}(Y)$ via

$$(B, \Psi) \mapsto g^*(B, \Psi) = (B - g^{-1}dg, g\Psi). \tag{3.1}$$

It is straightforward to check that the Seiberg-Witten map SW_3 is gauge equivariant (where gauge transformations act trivially on $\Omega^1(Y; i\mathbb{R})$). In particular, the space of solutions to the Seiberg-Witten equations is gauge-invariant.

The gauge group decomposes into a variety of important subgroups, which will be important for the various kinds of gauge fixing we will be doing. First, observe that $\pi_0(\mathcal{G})$, the number of connected components of $\mathcal{G}$, satisfies

$$\pi_0(\mathcal{G}) \cong H^1(Y; 2\pi i\mathbb{Z}). \tag{3.2}$$

The correspondence (3.2) is given by

$$g \mapsto [g^{-1}dg], \tag{3.3}$$

where the latter denotes the cohomology class of the closed 1-form $g^{-1}dg$. Among subgroups of the gauge group, one usually considers the group of harmonic gauge transformations, i.e., gauge transformations such that $g^{-1}dg \in \ker d^*$. However, on a manifold with boundary, $\ker(d + d^*)$ is infinite dimensional and we need to impose some boundary conditions.

On a manifold with boundary, Hodge theory tells us that we can make the following identifications between cohomology classes and harmonic forms with the appropriate boundary conditions[4]

$$H^1(Y; \mathbb{R}) \cong \{\alpha \in \Omega^1(Y) : da = d^*a = 0, *a|_\Sigma = 0\} \tag{3.4}$$
$$H^1(Y, \Sigma; \mathbb{R}) \cong \{\alpha \in \Omega^1(Y) : da = d^*a = 0, a|_\Sigma = 0.\}. \tag{3.5}$$

In fact, we have two different Hodge decompositions, given by

$$\Omega^1(Y) = \operatorname{im} d \oplus \operatorname{im} * d_n \oplus H^1(Y; \mathbb{R}) \tag{3.6}$$
$$= \operatorname{im} d_t \oplus \operatorname{im} * d \oplus H^1(Y, \Sigma; \mathbb{R}). \tag{3.7}$$

where

$$d_n : \{a \in \Omega^1(Y) : *a|_\Sigma = 0\} \to \Omega^2(Y) \tag{3.8}$$
$$d_t : \{\alpha \in \Omega^0(Y) : \alpha|_\Sigma = 0\} \to \Omega^1(Y). \tag{3.9}$$

Any gauge transformation g in the identity component of the gauge group $\mathcal{G}_{\mathrm{id}}(Y)$ lifts to the universal cover of S^1 and so it can be expressed as $g = e^\xi$ for some $\xi \in \Omega^0(Y; i\mathbb{R})$. For such g, we have $g^{-1}dg = d\xi$, and thus we see that $\mathcal{G}/\mathcal{G}_{\mathrm{id}}$ is isomorphic to the integer lattice inside $\ker d/\operatorname{im} d$, which establishes the correspondence (3.2). Corresponding to the two cohomology groups (3.4) and (3.5), we can consider the following two subgroups of the

[4] For a differential form a over a manifold X with boundary, $a|_{\partial X}$ always denotes the differential form on ∂X obtained via the restriction of those components tangential to ∂X. Otherwise, given a section u of a general vector bundle over X, $u|_X$ denotes the restriction of u to the boundary, which therefore has values in the bundle restricted to the boundary. This clash of notation should not cause confusion since it will always be clear which restriction map we are using based on the context.

harmonic gauge transformations

$$\mathcal{G}_{h,n}(Y) = \{g \in \mathcal{G} : g^{-1}dg \in \ker d^*, *dg|_\Sigma = 0\} \tag{3.10}$$
$$\mathcal{G}_{h,\partial}(Y) = \{g \in \mathcal{G} : g^{-1}dg \in \ker d^*, g|_\Sigma = 1\}. \tag{3.11}$$

The group (3.10) is isomorphic to $S^1 \times H^1(Y;\mathbb{Z})$, where the S^1 factor accounts for constant gauge transformations, and the group (3.11) is isomorphic to the integer lattice $H^1(Y,\Sigma;\mathbb{Z})$ inside (3.5).

Next, we have the subgroup

$$\mathcal{G}_\perp(Y) = \{e^\xi \in \mathcal{G}_{\mathrm{id}} : \int_Y \xi = 0\}.$$

Thus, identifying constant gauge transformations with S^1, we have the decompositions

$$\mathcal{G}_{\mathrm{id}}(Y) = S^1 \times \mathcal{G}_\perp(Y)$$
$$\mathcal{G}(Y) = \mathcal{G}_{h,n}(Y) \times \mathcal{G}_\perp(Y).$$

We have the following additional subgroups of the gauge group consisting of gauge transformations whose restriction to the boundary is the identity:

$$\mathcal{G}_\partial(Y) = \{g \in \mathcal{G}(Y) : g|_\Sigma = 1\} \tag{3.12}$$
$$\mathcal{G}_{\mathrm{id},\partial}(Y) = \mathcal{G}_{\mathrm{id}}(Y) \cap \mathcal{G}_\partial(Y) \tag{3.13}$$

Thus, we have

$$\mathcal{G}_\partial(Y) = \mathcal{G}_{h,\partial}(Y) \times \mathcal{G}_{\mathrm{id},\partial}(Y) \tag{3.14}$$

and

$$T_{\mathrm{id}}\mathcal{G}_{\mathrm{id},\partial}(Y) = \{\xi \in \Omega^0(Y; i\mathbb{R}) : \xi|_\Sigma = 0\}. \tag{3.15}$$

Since we consider the completion of our configuration spaces in Besov topologies, we must do so for the gauge groups as well. Thus, let $\mathcal{G}^{s,p}(Y)$ denote the completion of $\mathcal{G}(Y)$ in $B^{s,p}(Y)$ and similarly for the other gauge groups.

Lemma 3.1 *For $s > 3/p$, the $B^{s,p}(Y)$ completions of $\mathcal{G}(Y)$ and its subgroups are Banach Lie groups. If in addition $s > 1/2$, these groups act smoothly on $\mathfrak{C}^{s-1,p}(Y)$.*

Proof For $s > 3/p$, the multiplication theorem, Theorem 13.18, implies $B^{s,p}(Y)$ is a Banach algebra. Thus, $\mathcal{G}^{s,p}(Y)$ is closed under multiplication and has a smooth exponential map. The second statement follows from (3.1), Theorem 13.18, and the fact that $d : B^{s,p}(Y) \to B^{s-1,p}\Omega^1(Y)$ for all $s \in \mathbb{R}$ by Corollary 13.14. Here the requirement $s > 1/2$ comes from the fact that we need $s + (s-1) > 0$ in Theorem 13.18. □

Fix a smooth reference connection B_{ref}. From this, we obtain the Coulomb slice and Coulomb-Neumann slice through B^{ref}, given by

$$\mathfrak{C}^{s,p}_C(Y) = \{(B,\Psi) \in \mathfrak{C}^{s,p}(Y) : d^*(B - B^{\mathrm{ref}}) = 0\} \tag{3.16}$$
$$\mathfrak{C}^{s,p}_{C_n}(Y) = \{(B,\Psi) \in \mathfrak{C}^{s,p} : d^*(B - B^{\mathrm{ref}}) = 0, *(B - B^{\mathrm{ref}})|_\Sigma = 0\}, \qquad (s > 1/p) \tag{3.17}$$

respectively. The next lemma tells us that we can find gauge transformations which place any configuration into either of the above slices.

Lemma 3.2 *Let* $s+1 > \max(3/p, 1/2)$*. The action of the gauge group gives us the following decompositions of the configuration space:*

(i) We have[5]

$$\mathfrak{C}^{s,p}(Y) = \mathcal{G}^{s+1,p}_{\mathrm{id},\partial}(Y) \times \mathfrak{C}^{s,p}_{C}(Y). \tag{3.18}$$

(ii) Suppose in addition $s > 1/p$*. Then we have*

$$\mathfrak{C}^{s,p}(Y) = \mathcal{G}^{s+1,p}_{\perp}(Y) \times \mathfrak{C}^{s,p}_{C_n}(Y). \tag{3.19}$$

Proof (i) Since $s+1 > \max(3/p, 1/2)$, the previous lemma implies $\mathcal{G}^{s+1}_{\mathrm{id},\partial}(Y)$ is a Banach Lie group and it acts on $\mathfrak{C}^{s,p}(Y)$. If $u = e^{\xi} \in G^{s+1,p}_{\mathrm{id},\partial}$ puts a configuration $(B_{\mathrm{ref}} + b, \Psi)$ into the Coulomb slice through B_{ref}, then ξ satisfies

$$\begin{cases} \Delta\xi &= d^*b \in B^{s-1,p}(Y; i\mathbb{R}), \\ \xi|_{\Sigma} &= 0. \end{cases} \tag{3.20}$$

The Dirichlet Laplacian is an elliptic boundary value problem and since $s+1 > 1/p$, we may apply Corollary 15.22, which shows that we have an elliptic estimate

$$\|\xi\|_{B^{s+1,p}} \leq C(\|\Delta\xi\|_{B^{s,p}} + \|\xi\|_{B^{s,p}})$$

for ξ satisfying (3.20). A standard computation shows that the kernel and cokernel of the Dirichlet Laplacian is zero, and so we have existence and uniqueness for the Dirichlet problem. This implies the decomposition.

(ii) The analysis is the same, only now we have a homogeneous Neumann Laplacian problem for ξ:

$$\begin{cases} \Delta\xi &= d^*b \in B^{s-1,p}(Y; i\mathbb{R}) \\ *d\xi|_{\Sigma} &= *b|_{\Sigma} \in B^{s-1/p,p}\Omega^2(\Sigma; i\mathbb{R}). \end{cases} \tag{3.21}$$

Since the Neumann Laplacian is an elliptic boundary value problem, we can apply Corollary 15.22 again. The inhomogeneous Neumann problem $\Delta\xi = f$ and $\partial_\nu \xi = g$ has a solution if and only if $\int_Y f + \int_\Sigma g = 0$, and this solution is unique up to constant functions. Since we always have $\int_Y d^*b + \int_\Sigma *b = 0$, then (3.21) has a unique solution $\xi \in B^{s+1,p}(Y)$ subject to $\int_Y \xi = 0$. The decomposition now follows. □

In light of Lemma 3.2, we can regard the quotient of $\mathfrak{C}^{s,p}(Y)$ by the gauge groups $\mathcal{G}^{s+1,p}_{\mathrm{id},\partial}$ and $\mathcal{G}^{s+1,p}_{\perp}$ as subspaces of $\mathfrak{C}^{s,p}(Y)$, namely, those configurations in Coulomb and Coulomb-Neumann gauge with respect to B_{ref}.

[5]The direct products appearing in (3.18) and (3.19) mean that the gauge group factor acts freely on the subspace appearing in the second factor so that the space on the left is equal to the resulting orbit space obtained from the right-hand side.

Remark 3.3 In gauge theory, one usually also considers the quotient of the configuration space by the entire gauge group. In our case (which is typical) the quotient space is singular since different elements of the configuration space have different stabilizers. Namely, if $(B,\Psi) \in \mathfrak{C}(Y)$ is such that $\Psi \not\equiv 0$, then it has trivial stabilizer, whereas if $\Psi \equiv 0$, then it has stabilizer S^1, the constant gauge transformations. In the former case, such a configuration is said to be *irreducible*, otherwise it is *reducible*. We will not need to consider the quotient space by the entire gauge group in Part I, and we will only need to consider the decompositions in Lemma 3.2.

3.2 Decompositions of the Tangent Space

The action of the gauge group on the configuration space induces a decomposition of the tangent space to a configuration (B,Ψ) into the subspace tangent to the gauge orbit through (B,Ψ) and its orthogonal complement. More precisely, let

$$\mathcal{T}_{(B,\Psi)} := T_{(B,\Psi)}\mathfrak{C}(Y) = \Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \tag{3.22}$$

be the smooth tangent space to a smooth configuration (B,Ψ). Define the operator

$$\begin{aligned} \mathbf{d}_{(B,\Psi)} : \Omega^0(Y;i\mathbb{R}) &\to \mathcal{T}_{(B,\Psi)} \\ \xi &\mapsto (-d\xi, \xi\Psi), \end{aligned} \tag{3.23}$$

and let

$$\mathcal{J}_{(B,\Psi)} := \operatorname{im} \mathbf{d}_{(B,\Psi)} \subset \mathcal{T}_{(B,\Psi)} \tag{3.24}$$

be its image. Then observe that $\mathcal{J}_{(B,\Psi)}$ is the tangent space to the gauge orbit at (B,Ψ). Indeed, this follows from differentiating the action (3.1) at the identity. We also have the adjoint operator

$$\begin{aligned} \mathbf{d}^*_{(B,\Psi)} : \mathcal{T}_{(B,\Psi)} &\to \Omega^0(Y;i\mathbb{R}) \\ (b,\psi) &\mapsto -d^*b + i\mathrm{Re}\,(i\Psi,\psi), \end{aligned} \tag{3.25}$$

and we define the subspace

$$\mathcal{K}_{(B,\Psi)} := \ker \mathbf{d}^*_{(B,\Psi)} \subset \mathcal{T}_{(B,\Psi)}. \tag{3.26}$$

On a closed manifold, $\mathcal{K}_{(B,\Psi)}$ is the L^2 orthogonal complement of $\operatorname{im}\mathbf{d}_{(B,\Psi)}$. In this case, the orthogonal decomposition of $\mathcal{T}_{(B,\Psi)}$ into the spaces $\mathcal{J}_{(B,\Psi)}$ and $\mathcal{K}_{(B,\Psi)}$ plays a fundamental role in the analysis of [21]. In our case, since we have a boundary, we will impose various boundary conditions on these spaces, and the resulting spaces will play a very important role for us too. Moreover, we will take the appropriate Besov completions of these spaces.

Thus, let $(B,\Psi) \in \mathfrak{C}^{t,q}(Y)$ be any configuration of regularity $B^{t,q}(Y)$, where $t \in \mathbb{R}$ and $q \geq 2$. For $s \in \mathbb{R}$ and $p \geq 2$, let

$$\mathcal{T}^{s,p}_{(B,\Psi)} := B^{s,p}(\Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S})) \tag{3.27}$$

be the Besov closure of $\mathcal{T}_{(B,\Psi)}$. It is independent of (B,Ψ) and is equal to the tangent space

$T_{(B,\Psi)}\mathfrak{C}^{s,p}(Y)$ when $(s,p) = (t,q)$. So long as we have bounded multiplication maps

$$B^{t,q}(Y) \times B^{s+1,p}(Y) \to B^{s,p}(Y) \tag{3.28}$$
$$B^{t,q}(Y) \times B^{s,p}(Y) \to B^{s-1,p}(Y), \tag{3.29}$$

then we can define maps

$$\begin{aligned} \mathbf{d}_{(B,\Psi)} : B^{s+1,p}\Omega^0(Y; i\mathbb{R}) &\to \mathcal{T}^{s,p}_{(B,\Psi)} \\ \xi &\mapsto (-d\xi, \xi\Psi), \\ \mathbf{d}^*_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} &\to B^{s-1,p}\Omega^0(Y; i\mathbb{R}) \\ (b,\psi) &\mapsto -d^*b + i\mathrm{Re}\,(i\Psi, \psi), \end{aligned}$$

respectively. In particular, if $(t,q) = (s,p)$ and $s > 3/p$, then by Theorem 13.18, the multiplications (3.28) and (3.29) are bounded.

Thus, when (3.28) and (3.29) hold, define the following subspaces of $\mathcal{T}^{s,p}_{(B,\Psi)}$:

$$\mathcal{J}^{s,p}_{(B,\Psi)} = \mathrm{im}\left(\mathbf{d}_{(B,\Psi)} : B^{s+1,p}\Omega^0(Y; i\mathbb{R}) \to \mathcal{T}^{s,p}_{(B,\Psi)}\right) \tag{3.30}$$
$$\mathcal{J}^{s,p}_{(B,\Psi),\perp} = \{(-d\xi, \xi\Psi) \in \mathcal{J}^{s,p}_{(B,\Psi)} : \int_Y \xi = 0\} \tag{3.31}$$
$$\mathcal{J}^{s,p}_{(B,\Psi),t} = \{(-d\xi, \xi\Psi) \in \mathcal{J}^{s,p}_{(B,\Psi)} : \xi|_\Sigma = 0\} \tag{3.32}$$
$$\mathcal{K}^{s,p}_{(B,\Psi)} = \ker\left(\mathbf{d}^*_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to B^{s-1,p}\Omega^0(Y; i\mathbb{R})\right) \tag{3.33}$$
$$\mathcal{K}^{s,p}_{(B,\Psi),n} = \{(b,\psi) \in \mathcal{K}^{s,p}_{(B,\Psi)} : *b|_\Sigma = 0\}. \tag{3.34}$$

Observe that when $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, then $\mathcal{J}^{s,p}_{(B,\Psi)}$, $\mathcal{J}^{s,p}_{\perp}$, $\mathcal{J}^{s,p}_{(B,\Psi),t}$ are the tangent spaces to the gauge orbit of (B,Ψ) in $\mathfrak{C}^{s,p}$ determined by the gauge groups $\mathcal{G}^{s+1,p}(Y)$, $\mathcal{G}^{s+1,p}_{\perp}(Y)$, and $\mathcal{G}^{s+1,p}_{\partial}(Y)$, respectively. Note that the subscript t appearing in $\mathcal{J}^{s,p}_{(B,\Psi),t}$ is a label to denote that the (tangential) restriction of ξ to the boundary vanishes; it is not to be confused with a real parameter. This is consistent with the notation used in (3.9). Likewise, the subscript n appearing in $\mathcal{K}^{s,p}_{(B,\Psi),n}$ and (3.8) denotes that the elements belonging to these spaces have normal components for their 1-form parts equal to zero on the boundary. We also have the linear Coulomb and Coulomb-Neumann slices:

$$\mathcal{C}^{s,p}_{(B,\Psi)} = \{(b,\psi) \in \mathcal{T}^{s,p}_{(B,\Psi)} : d^*b = 0\} \tag{3.35}$$
$$\mathcal{C}^{s,p}_{(B,\Psi),n} = \{(b,\psi) \in \mathcal{T}^{s,p}_{(B,\Psi)} : d^*b = 0, *b|_\Sigma = 0\}. \tag{3.36}$$

The following lemma is essentially the linear version of Lemma 3.2. The statement is only mildly more technical in that one may consider the basepoint (B,Ψ) and the tangent space $\mathcal{T}_{(B,\Psi)}$ in different topologies. We do this because we will need to consider topologies on $\mathcal{T}_{(B,\Psi)}$ that are weaker than the regularity of (B,Ψ), which occurs, for example, when we apply differential operators to elements of $\mathcal{T}^{s,p}_{(B,\Psi)}$ when $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, thereby obtaining spaces such as $\mathcal{T}^{s-1,p}_{(B,\Psi)}$. These spaces and their decompositions will become important for us in the next section, when we study the linearized Seiberg-Witten equations and try to recast them in a form in which they become elliptic.

Lemma 3.4 *Let $s+1>1/p$ and let $(B,\Psi)\in\mathfrak{C}^{t,q}(Y)$, where $t>3/q$, $q\geq 2$ are such that (3.28) and (3.29) hold. In particular, if $q=p$, then we need $t\geq s$ and $t>\max(-s,3/p)$.*

(i) We have the following decompositions:

$$\mathcal{T}^{s,p}_{(B,\Psi)}=\mathcal{J}^{s,p}_{(B,\Psi),t}\oplus\mathcal{K}^{s,p}_{(B,\Psi)} \tag{3.37}$$

$$\mathcal{T}^{s,p}_{(B,\Psi)}=\mathcal{J}^{s,p}_{(B,\Psi),t}\oplus\mathcal{C}^{s,p}_{(B,\Psi)}. \tag{3.38}$$

(ii) If in addition $s>1/p$, then

$$\mathcal{T}^{s,p}_{(B,\Psi)}=\mathcal{J}^{s,p}_{(B,\Psi),\perp}\oplus\mathcal{C}^{s,p}_{(B,\Psi),n}. \tag{3.39}$$

If $\Psi\not\equiv 0$, then furthermore

$$\mathcal{T}^{s,p}_{(B,\Psi)}=\mathcal{J}^{s,p}_{(B,\Psi)}\oplus\mathcal{K}^{s,p}_{(B,\Psi),n}. \tag{3.40}$$

Proof We first prove (3.37). Given $(b,\psi)\in\mathcal{T}^{s,p}_{(B,\Psi)}$, consider the boundary value problem

$$\begin{cases}\Delta_{(B,\Psi)}\xi &= f\in B^{s-1,p}(Y;i\mathbb{R})\\ \xi|_\Sigma &=0,\end{cases} \tag{3.41}$$

where $f=\mathbf{d}^*_{(B,\Psi)}b$ and

$$\Delta_{(B,\Psi)}:=\mathbf{d}^*_{(B,\Psi)}\mathbf{d}_{(B,\Psi)}=\Delta+|\Psi|^2. \tag{3.42}$$

We have $\mathbf{d}^*_{(B,\Psi)}b\in B^{s-1,p}(Y;i\mathbb{R})$ since we have a bounded multiplication $B^{t,p}(Y)\times B^{s,p}(Y)\to B^{s-1,p}(Y)$ by the hypotheses. Likewise, since we have a bounded map $B^{t,p}(Y)\times B^{s+1,p}(Y)\to B^{s,p}(Y)$, we see that multiplication by $|\Psi|^2\in B^{t,p}(Y)$ is a compact perturbation of $\Delta: B^{s+1,p}(Y)\to B^{s-1,p}(Y)$. Thus, the Dirichlet boundary value problem (3.41) is Fredholm for $s+1>1/p$ (where the requirement on s is so that Dirichlet boundary conditions make sense, cf. Corollary 15.22). Moreover, since $|\Psi|^2$ is a positive multiplication operator, a simple computation shows the existence and uniqueness of (3.41). Indeed, if $\Delta\alpha=-|\Psi|^2\alpha$ and $\alpha|_\Sigma=0$, then repeated elliptic boostrapping for the inhomogeneous Dirichlet Laplacian shows that $\alpha\in B^{t+2,q}(Y)\subseteq B^{2,2}(Y)$ since $t>0$ and $q\geq 2$. Then

$$0=\int_Y(\Delta_{(B,\Psi)}\alpha,\alpha)=\|\nabla\alpha\|^2_{L^2(Y)}+\|\Psi\alpha\|^2_{L^2(Y)},$$

which implies α is constant. Hence, $\alpha=0$ since $\alpha|_\Sigma=0$. Thus, (3.41) has no kernel and since the adjoint problem of (3.41) is itself, we see that (3.41) has no cokernel as well. Thus, the existence and uniqueness of (3.41) is established. Let $\Delta^{-1}_{(B,\Psi),t}$ denote the solution map of (3.41). We have shown that $\Delta^{-1}_{(B,\Psi),t}: B^{s-1,p}(Y)\to B^{s+1,p}(Y)$ is bounded. The projection onto $\mathcal{J}^{s,p}_{(B,\Psi),t}$ through $\mathcal{K}^{s,p}_{(B,\Psi)}$ is now seen to be given by

$$\Pi_{\mathcal{J}^{s,p}_{(B,\Psi),t}}=\mathbf{d}_{(B,\Psi)}\Delta^{-1}_{(B,\Psi),t}\mathbf{d}^*_{(B,\Psi)} \tag{3.43}$$

and it is bounded on $\mathcal{T}^{s,p}_{(B,\Psi)}$ since $\mathbf{d}_{(B,\Psi)} : B^{s+1,p}\Omega^0(Y;i\mathbb{R}) \to B^{s,p}\Omega^1(Y;i\mathbb{R})$ is bounded. This gives us the decomposition (3.37). Similarly, we get the decomposition (3.38) if we replace $\Delta_{(B,\Psi)}$ with Δ in the above.

For (ii), if we consider the inhomogeneous Neumann problem for $\Delta_{(B,\Psi)}$ instead of the Dirichlet problem, proceeding as above yields (3.40), since when $\Psi \not\equiv 0$, a similar computation shows that we get existence and uniqueness. Here, we need $s > 1/p$ so that $s+1 > 1+1/p$ and the relevant Neumann boundary condition makes sense. Similarly, considering the inhomogeneous Neumann problem for Δ yields (3.39). □

For any $s,t \in \mathbb{R}$, we can define the Banach bundle

$$\mathcal{T}^{s,p}(Y) \to \mathfrak{C}^{t,p}(Y) \tag{3.44}$$

whose fiber over every $(B,\Psi) \in \mathfrak{C}^{t,p}(Y)$ is the Banach space $\mathcal{T}^{s,p}_{(B,\Psi)}$. Of course, all the $\mathcal{T}^{s,p}_{(B,\Psi)}$ are identical, so the bundle (3.44) is trivial. If $s = t$, then (3.44) is the tangent bundle of $\mathfrak{C}^{t,p}(Y)$. If s,t satisfy the hypotheses of the previous lemma, decomposing each fiber $\mathcal{T}^{s,p}_{(B,\Psi)}$ according to the decomposition (3.37) defines us Banach subbundles of $\mathcal{T}^{s,p}(Y)$. This is the content of the below proposition, where we specialize to a range of parameters relevant to the situations we will encounter later, e.g., see Lemma 4.1.

Proposition 3.5 *Let $s > 3/p$. If $\max(-s,-1+1/p) < s' \le s$, then the Banach bundles*

$$\mathcal{J}^{s',p}_t(Y) \to \mathfrak{C}^{s,p}(Y)$$
$$\mathcal{K}^{s',p}(Y) \to \mathfrak{C}^{s,p}(Y),$$

whose fibers over $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$ are $\mathcal{J}^{s',p}_{(B,\Psi),t}$ and $\mathcal{K}^{s',p}_{(B,\Psi)}$, respectively, are complementary subbundles of $\mathcal{T}^{s',p}(Y)$.

Proof The restrictions on s and s' ensure that we can apply Lemma 3.4. From this, one has to check that the resulting decomposition

$$\mathcal{T}^{s',p'}_{(B,\Psi)} = \mathcal{J}^{s',p'}_{(B,\Psi),t} \oplus \mathcal{K}^{s',p'}_{(B,\Psi)}$$

varies continuously with $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. For this, it suffices to show that the projection $\Pi_{\mathcal{J}^{s',p}_{(B,\Psi),t}}$ given by (3.43), with range $\mathcal{J}^{s',p}_{(B,\Psi),t}$ and kernel $\mathcal{K}^{s',p}_{(B,\Psi)}$, varies continuously with $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. Once we prove that $\mathcal{J}^{s',p}_t(Y)$ is a subbundle, it automatically follows that $\mathcal{K}^{s',p}(Y)$ is a (complementary) subbundle, since then the complementary projection

$$\Pi_{\mathcal{K}^{s',p}_{(B,\Psi)}} = 1 - \Pi_{\mathcal{J}^{s',p}_{(B,\Psi),t}} \tag{3.45}$$

onto $\mathcal{K}^{s',p}_{(B,\Psi)}$ varies continuously with (B,Ψ).

From the multiplication theorem, Theorem 13.18, since

$$\Delta_{(B,\Psi),t} : \{\xi \in B^{s'+1,p}(Y;i\mathbb{R}) : \xi|_\Sigma = 0\} \to B^{s'-1,p}(Y;i\mathbb{R})$$

varies continuously with $(B,\Psi) \in B^{s,p}(Y)$ and is an isomorphism for all (B,Ψ), its inverse $\Delta^{-1}_{(B,\Psi),t}$ also varies continuously. Likewise, $\mathbf{d}^*_{(B,\Psi)} : \mathcal{T}^{s',p}_{(B,\Psi)} \to B^{s'-1,p}\Omega^0(Y;i\mathbb{R})$ and $\mathbf{d}_{(B,\Psi)} : B^{s'+1,p}\Omega^0(Y,i\mathbb{R}) \to \mathcal{T}^{s',p}_{(B,\Psi)}$ vary continuously with $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. This establishes the required continuity of $\Pi_{\mathcal{J}^{s',p}_{(B,\Psi)}} = \mathbf{d}_{(B,\Psi)}\Delta^{-1}_{(B,\Psi),t}\mathbf{d}^*_{(B,\Psi)}$ with respect to (B,Ψ). □

The Banach bundle $\mathcal{K}^{s',p}(Y)$, with $s' = s-1$ will be used to establish transversality properties of the Seiberg-Witten map SW_3, see Theorem 4.2.

3.3 The Linearized Seiberg-Witten Equations

In this section, we study the linearization of the Seiberg-Witten map SW_3 to prove basic properties concerning the (formal) tangent space to our monopole spaces on Y and their behavior under restriction to the boundary. If the linearization of the Seiberg-Witten equations were elliptic, this would be quite straightforward from the analysis of elliptic boundary value problems, the relevant results of which are summarized in Part IV. However, because the Seiberg-Witten equations are gauge-invariant, its linearization is not elliptic and we have to do some finessing to account for the gauge-invariance. To do this, we make fundamental use of the subspaces and decompositions of the previous section.

Before we get started, let us note that our main theorem of this section, Theorem 3.13, proves a bit more than what is needed to prove our main theorems. Indeed, it is mostly phrased in such a way that the results of this section can be tied into the general framework of the pseudodifferential analysis of elliptic boundary value problems in Section 15.3 (see the discussion preceding Theorem 3.13). Moreover, some of the consequences of Theorem 3.13 will only be put to full use in Part III. Thus, the reader should regard this section as a general framework for studying the Hessian and augmented Hessian operators, (3.50) and (3.54), whose kernels are equal to the tangent spaces to $\mathfrak{M}$ and $\mathcal{M}$, respectively, via (3.51) and (3.55). Much of this framework consists in the construction of pseudodifferential type operators associated to the Hessian and augmented operators, namely the Calderon projection and Poisson operators, see Lemma 3.12 and Definition 3.14. For the augmented Hessian, an elliptic operator, these operators are defined as in Definition 15.15, and for the non-elliptic Hessian, they are defined by analogy in Definition 3.14. In a few words, the significance of these operators is that they relate the kernel of the (augmented) Hessian with the kernel's boundary values in a simple and uniform way across multiple topologies. This is what allows us to relate the tangent spaces to $\mathfrak{M}$ and $\mathcal{M}$ with the tangent spaces to $\mathcal{L}$, the latter being the boundary values of the kernels of the Hessian operators via (3.52). Unfortunately, the infinite dimensional nature of all spaces involved and the presence of multiple topologies makes the work we do quite technical. As a suggestion to the reader, it would be best to first absorb the main ideas of Section 15.3 and to understand the statements of Lemma 3.12 and Theorem 3.13 before plunging into the details.

Let

$$\mathcal{T} = \Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \tag{3.46}$$

be a fixed copy of the tangent space $\mathcal{T}_{(B,\Psi)} = T_{(B,\Psi)}\mathfrak{C}(Y)$ to any smooth configuration

$(B,\Psi) \in \mathfrak{C}(Y)$.[6] Thus, all the subspaces of $\mathcal{T}_{(B,\Psi)}$, namely $\mathcal{J}_{(B,\Psi)}$, $\mathcal{K}_{(B,\Psi)}$, and their associated subspaces defined in the previous section, may be regarded as subspaces of $\mathcal{T}$ that depend on a configuration $(B,\Psi) \in \mathfrak{C}(Y)$. We let

$$\mathcal{C} = \{(b,\psi) \in \mathcal{T} : d^*b = 0\} \tag{3.47}$$

denote the Coulomb-slice in $\mathcal{T}$. Likewise, let

$$\mathcal{T}_\Sigma = \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma) \tag{3.48}$$

denote a fixed copy of the tangent space to any smooth configuration of $\mathfrak{C}(\Sigma)$. The restriction map (2.5) on configuration spaces induces a restriction map on the tangent spaces

$$\begin{aligned} r_\Sigma &: \mathcal{T} \to \mathcal{T}_\Sigma \\ &(b,\psi) \mapsto (b|_\Sigma, \psi|_\Sigma). \end{aligned} \tag{3.49}$$

From (2.2), the linearization of the Seiberg-Witten map SW_3 at a configuration $(B,\Psi) \in \mathfrak{C}(Y)$ yields an operator

$$\begin{aligned} \mathcal{H}_{(B,\Psi)} &: \mathcal{T} \to \mathcal{T} \\ \mathcal{H}_{(B,\Psi)} &= \begin{pmatrix} *d & 2i\mathrm{Im}\,\rho^{-1}(\cdot\Psi^*)_0 \\ \rho(\cdot)\Psi & D_B \end{pmatrix} \end{aligned} \tag{3.50}$$

which acts on the tangent space $\mathcal{T}$ to (B,Ψ). We call the operator $\mathcal{H}_{(B,\Psi)}$ the *Hessian*.[7] The Hessian operator is a formally self-adjoint first order operator. For any monopole $(B,\Psi) \in SW_3^{-1}(0)$, we (formally) have that the tangent spaces to our monopole spaces $\mathfrak{M}$ and $\mathcal{L}$ are given by

$$T_{(B,\Psi)}\mathfrak{M} = \ker \mathcal{H}_{(B,\Psi)} \tag{3.51}$$

$$T_{r_\Sigma(B,\Psi)}\mathcal{L} = r_\Sigma(\ker \mathcal{H}_{(B,\Psi)}). \tag{3.52}$$

Indeed, this is just the linearization of (2.3) and (2.6). Thus, understanding $\mathfrak{M}$ and $\mathcal{L}$ at the linear level is the same as understanding the kernel of $\mathcal{H}_{(B,\Psi)}$.

Unfortunately, $\mathcal{H}_{(B,\Psi)}$ is not elliptic, which follows from a simple examination of its symbol. In fact, this nonellipticity follows a priori from the equivariance of the Seiberg-Witten map under gauge transformations. In particular, since the zero set of SW_3 is gauge-invariant, then the linearization $\mathcal{H}_{(B,\Psi)}$ at a monopole (B,Ψ) annihilates the entire tangent space to the gauge orbit at (B,Ψ), i.e., the subspace $\mathcal{J}_{(B,\Psi)} \subset \mathcal{T}$. Furthermore, even if we were to account for this gauge invariance by say, placing configurations in Coulomb-gauge, i.e., if we were instead to consider the operator $\mathcal{H}_{(B,\Psi)} \oplus d^* : \mathcal{T} \to \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R})$, we still would not have an elliptic operator in the usual sense.

[6] There is no real distinction between $\mathcal{T}$ and a particular tangent space $\mathcal{T}_{(B,\Psi)}$ to a configuration, since $\mathfrak{C}(Y)$ is an affine space. However, when we study the spaces $\mathfrak{M}$ and $\mathcal{M}$ as subsets of $\mathfrak{C}(Y)$ in Section 4, we will reintroduce base points when we have a particular tangent space in mind. For now, we drop basepoints to minimize notation.

[7] On a closed manifold, $\mathcal{H}_{(B,\Psi)}$ would in fact be the Hessian of the Chern-Simons-Dirac functional, see [21].

However, there is a simple remedy for this predicament. Following [21], the operator $\mathcal{H}_{(B,\Psi)}$ naturally embeds as a summand of an elliptic operator. Namely, if we enlarge the space $\mathcal{T}$ to the *augmented tangent space*

$$\widetilde{\mathcal{T}} := \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R}), \tag{3.53}$$

then we can consider the *augmented Hessian*[8]

$$\begin{aligned} \widetilde{\mathcal{H}}_{(B,\Psi)} &: \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} \\ \widetilde{\mathcal{H}}_{(B,\Psi)} &= \begin{pmatrix} \mathcal{H}_{(B,\Psi)} & -d \\ -d^* & 0 \end{pmatrix}. \end{aligned} \tag{3.54}$$

The augmented Hessian is a formally self-adjoint first order elliptic operator, as one can easily verify. This operator takes into account Coulomb gauge-fixing via the operator $d^* : \Omega^1(Y; i\mathbb{R}) \to \Omega^0(Y; i\mathbb{R})$, while ensuring ellipticity by adding in the adjoint operator $d : \Omega^0(Y; i\mathbb{R}) \to \Omega^1(Y; i\mathbb{R})$. The advantage of studying the operator $\widetilde{\mathcal{H}}_{(B,\Psi)}$ is that we may apply the pseudodifferential tools from Section 15.3 to understand the kernel of $\widetilde{\mathcal{H}}_{(B,\Psi)}$ and its boundary values. Moreover, have (formally) that

$$T_{(B,\Psi)}\mathcal{M} = \ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{T}}). \tag{3.55}$$

The space of boundary values for $\widetilde{\mathcal{T}}$ is the space

$$\widetilde{\mathcal{T}}_\Sigma := \mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}). \tag{3.56}$$

Indeed, one can see that $\widetilde{\mathcal{T}}|_\Sigma \cong \widetilde{\mathcal{T}}_\Sigma$ via the full restriction map $r : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}_\Sigma$ given by

$$\begin{aligned} r : \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \oplus \Omega^0(Y; i\mathbb{R}) &\to \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma) \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \\ (b, \psi, \alpha) &\mapsto (b|_\Sigma, \psi|_\Sigma, -b(\nu), \alpha|_\Sigma), \end{aligned} \tag{3.57}$$

where in (3.57), the term $b(\nu)$ denotes contraction of the 1-form b with the outward normal ν to Σ. Thus, the two copies of $\Omega^0(\Sigma; i\mathbb{R})$ in $\widetilde{\mathcal{T}}_\Sigma$ are meant to capture the normal component of $\Omega^1(Y; i\mathbb{R})$ and the trace of $\Omega^0(Y; i\mathbb{R})$ along boundary. The map $r_\Sigma : \mathcal{T} \to \mathcal{T}_\Sigma$ appears as the first factor of the map r, and it is the tangential part of the full restriction map. Since we can regard $\mathcal{T} \subset \widetilde{\mathcal{T}}$, then by restriction, the map r also maps $\mathcal{T}$ to $\widetilde{\mathcal{T}}_\Sigma$.

As usual, we can consider the Besov completions of all the spaces involved. Thus, we have the spaces

$$\mathcal{C}^{s,p}, \mathcal{T}^{s,p}, \ \widetilde{\mathcal{T}}^{s,p}, \ \mathcal{T}_\Sigma^{s,p}, \ \widetilde{\mathcal{T}}_\Sigma^{s,p}$$

which we use to denote the $B^{s,p}$ completions of their corresponding smooth counterparts. The restriction maps r_Σ and r extend to Besov completions in the usual way. We also have

[8]In [21], the operators $\mathbf{d}_{(B,\Psi)}$ and $\mathbf{d}^*_{(B,\Psi)}$ are used in the definition of $\widetilde{\mathcal{H}}_{(B,\Psi)}$ instead of $-d$ and $-d^*$, respectively. Our definition reflects the fact that we will work with Coulomb slices $\mathcal{C}_{(B,\Psi)}$ instead of the slices $\mathcal{K}_{(B,\Psi)}$ inside $\mathcal{T}$. The presence of the minus signs on $-d$ and $-d^*$ in $\widetilde{\mathcal{H}}_{(B,\Psi)}$ lies in the relationship between $\widetilde{\mathcal{H}}_{(B,\Psi)}$ and the linearization of the 4-dimensional Seiberg-Witten equations, see Part III. Thus, the augmented Hessian operator is not an ad-hoc extension of the Hessian operator, but is tied to the underlying geometry of the problem.

the spaces $\mathcal{J}^{s,p}_{(B,\Psi)}$, $\mathcal{K}^{s,p}_{(B,\Psi)}$, and their subspaces from the previous section, which we may all regard as subspaces of $\mathcal{T}^{s,p}$.

The plan for the rest of this section is as follows. First, we investigate the kernel of the elliptic operator $\widetilde{\mathcal{H}}_{(B,\Psi)}$. We do this first for smooth (B,Ψ), in which case the tools from Section 15.3 apply, and then we consider nonsmooth (B,Ψ), in which case modifications must be made. Here, one has to keep track of the function space arithmetic rather carefully. Next, we will relate the kernel of $\widetilde{\mathcal{H}}_{(B,\Psi)}$ to the kernel of $\mathcal{H}_{(B,\Psi)}$ and see how these spaces behave under the restriction maps r and r_Σ, respectively. For this, we place these results under the conceptual framework of Section 15.3 by way of using the Calderon projection and Poisson operator associated to an elliptic operator. For the Hessian $\mathcal{H}_{(B,\Psi)}$, the main technical issue here is its non-ellipticity (i.e. gauge-invariance). The results of our analysis are summarized in the main theorem of this section, Theorem 3.13.[9]

$$
\begin{array}{ccl}
\widetilde{\mathcal{T}} & \xrightarrow{\widetilde{\mathcal{H}}_{(B,\Psi)}} & \widetilde{\mathcal{T}} = \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R}) \\
\uparrow & & \uparrow \\
\mathcal{T} & \xrightarrow{\mathcal{H}_{(B,\Psi)}} & \mathcal{T} = \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S})
\end{array}
$$

$$
\begin{array}{ccl}
\widetilde{\mathcal{T}} & \xrightarrow{r} & \widetilde{\mathcal{T}}_\Sigma = \mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \\
\uparrow & & \uparrow \\
\mathcal{T} & \xrightarrow{r_\Sigma} & \mathcal{T}_\Sigma = \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma)
\end{array}
\tag{3.58}
$$

In studying the augmented Hessian operators $\widetilde{\mathcal{H}}_{(B,\Psi)}$ for smooth $(B,\Psi) \in \mathfrak{C}(Y)$, observe that they all differ by bounded zeroth order operators. Indeed, if we write $(b,\psi) = (B_1,\Psi_1) - (B_0,\Psi_0)$, then

$$\widetilde{\mathcal{H}}_{(B_1,\Psi_1)} - \widetilde{\mathcal{H}}_{(B_0,\Psi_0)} = (b,\psi)\#$$

where $(b,\psi)\#$ is the multiplication operator given by

$$
\begin{aligned}
(b,\psi)\# : \mathcal{T} &\to \mathcal{T} \\
(b',\psi') &\mapsto (2i\mathrm{Im}\,\rho^{-1}(\psi'\psi^*)_0, \rho(b)\psi').
\end{aligned}
\tag{3.59}
$$

In general, we will use # to denote any kind of pointwise multiplication map. Let B_{ref} be

[9] The complexity of the function space arithmetic in this section can minimized if one does not care about the symplectic properties of the spaces involved, namely, the Lagrangian properties in Lemma 3.11 and Theorem 3.13(i). See Remark 3.10.

our fixed smooth reference connection. Define

$$\begin{aligned}\widetilde{\mathcal{H}}_0 &:= \widetilde{\mathcal{H}}_{(B_{\mathrm{ref}},0)} \\ &= D_{\mathrm{dgc}} \oplus D_{B_{\mathrm{ref}}} \end{aligned} \tag{3.60}$$

where $D_{B_{\mathrm{ref}}} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ is the Dirac operator on spinors determined by B_{ref} and D_{dgc} is the div-grad curl operator

$$D_{\mathrm{dgc}} := \begin{pmatrix} *d & -d^* \\ -d & 0 \end{pmatrix} : \Omega^1(Y; i\mathbb{R}) \oplus \Omega^0(Y; i\mathbb{R}) \circlearrowleft \tag{3.61}$$

The operator D_{dgc} is also a Dirac operator. Thus, the operator $\widetilde{\mathcal{H}}_0$ is a Dirac operator and every other $\widetilde{\mathcal{H}}_{(B,\Psi)}$ is a zeroth order perturbation of $\widetilde{\mathcal{H}}_0$. Our first objective therefore is to understand the operator $\widetilde{\mathcal{H}}_0$.

Let us quickly review some basic properties about general Dirac operators. Let D be any Dirac operator acting on sections $\Gamma(E)$ of a Clifford bundle E over Y endowed with a connection compatible with the Clifford multiplication. Here, by a Dirac operator, we mean any operator equal to "the" Dirac operator on E (the operator determined by the Clifford multiplication and compatible connection) plus any zeroth order symmetric operator. Let $(\cdot,\cdot)$ denote the (real or Hermitian) inner product on E. Working in a collar neighborhood of $[0,\epsilon] \times \Sigma$ of the boundary, where $t \in [0,\epsilon]$ is the inward normal coordinate, we can identify $\Gamma(E|_{[0,\epsilon]\times\Sigma})$ with $\Gamma([0,\epsilon], \Gamma(E_\Sigma))$, the space of t-dependent sections with values in $\Gamma(E_\Sigma)$. Under this idenfication, we can write any Dirac operator D as

$$\mathsf{D} = \mathsf{J}_t \left(\frac{d}{dt} + \mathsf{B}_t + \mathsf{C}_t \right), \tag{3.62}$$

where J_t, B_t, and C_t are t-dependent operators acting on $\Gamma(E_\Sigma)$. The operator J_t is a skew-symmetric bundle automorphism satisfying $\mathsf{J}_t^2 = -\mathrm{id}$, the operator B_t is a first order elliptic self-adjoint operator, and C_t is a zeroth order bundle endomorphism.

Definition 3.6 We call $\mathsf{B}_0 : \Gamma(E_\Sigma) \to \Gamma(E_\Sigma)$ the *tangential boundary operator* associated to D.

Observe that the above definition is only well-defined up to a symmetric zeroth order term. By abuse of terminology, we may also refer to the family of operators B_t in (3.62) as tangential boundary operators as well.

The significance of the decomposition (3.62) is that the space of boundary values of the kernel of D is, up to a compact error, determined by the operator B_0. More precisely, we have the following picture. Since B_0 is a first order self-adjoint elliptic operator, the space $\Gamma(E_\Sigma)$ decomposes as

$$\Gamma(E_\Sigma) = \mathcal{Z}^+_{\mathsf{B}_0} \oplus \mathcal{Z}^-_{\mathsf{B}_0} \oplus \mathcal{Z}^0_{\mathsf{B}_0}, \tag{3.63}$$

the positive, negative, and zero spectral subspaces of B_0, respectively. Moreover, since the projections onto these subspaces are given by pseudodifferential operators, we get a corresponding decomposition on the Besov space completion:

$$B^{s,p}(E_\Sigma) = B^{s,p}\mathcal{Z}^+_{\mathsf{B}_0} \oplus B^{s,p}\mathcal{Z}^-_{\mathsf{B}_0} \oplus B^{s,p}\mathcal{Z}^0_{\mathsf{B}_0}, \tag{3.64}$$

for all $s \in \mathbb{R}$ and $1 < p < \infty$. If we let $\mathsf{D} : B^{s,p}(E) \to B^{s-1,p}(E)$, then we can consider the boundary values of its kernel $r(\ker \mathsf{D}) \subset B^{s-1/p,p}(E_\Sigma)$. Then what we have is that the spaces $r(\ker \mathsf{D})$ and $B^{s-1/p,p}\mathcal{Z}^+_{\mathsf{B}_0}$ are *commensurate*, that is, they differ by a compact perturbation[10] (see Definition 18.2). Furthermore, from Proposition 15.18, we have that $r(\ker D)$ is a Lagrangian subspace of the boundary data space $B^{s-1/p,p}(E_\Sigma)$, where the symplectic form on the Banach space $B^{s-1/p,p}(E_\Sigma)$ is given by Green's formula[11] for D:

$$\int_\Sigma \mathrm{Re}\,(u, -\mathsf{J}_0 v) = \mathrm{Re}\,(u, \mathsf{D}v)_{L^2(Y)} - \mathrm{Re}\,(\mathsf{D}u, v)_{L^2(Y)}. \tag{3.66}$$

Summarizing, we have

Lemma 3.7 *The Cauchy data space* $r(\ker \mathsf{D}) \subset B^{s-1/p,p}(E_\Sigma)$ *is a Lagrangian subspace commensurate with* $B^{s-1/p,p}\mathcal{Z}^+_{\mathsf{B}_0}$. *Furthermore, for* $s > 1/p$, *the space* $\ker \mathsf{D}$ *is complemented in* $B^{s,p}(E)$.

The last statement follows from Corollary 15.17. Thus, while $r(\ker \mathsf{D})$ is a space determined by the entire operator D on Y, it is "close" to the subspace $B^{s-1/p,p}\mathcal{Z}^+_{\mathsf{B}_0}$, which is completely determined on the boundary.

Let us now apply the above general framework to our Hessian operators. Let B denote the tangential boundary operator for $\widetilde{\mathcal{H}}_0$. By (3.60), B splits as a direct sum of the tangential boundary operators

$$\begin{aligned} \mathsf{B}_{\mathrm{dgc}} : \Omega^1(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \circlearrowleft \\ \mathsf{B}_{\mathcal{S}} : \Gamma(\mathcal{S}_\Sigma) \circlearrowleft, \end{aligned}$$

for D_{dgc} and $D_{B_{\mathrm{ref}}}$, respectively. For the div-grad-curl operator D_{dgc}, we can compute the tangential boundary operator and its spectrum rather explicitly. As before, we work inside a collar neighborhood $[0, \epsilon] \times \Sigma$ of the boundary of Y, with the inward normal coordinate given by $t \in [0, \epsilon]$, and we choose coordinates so that the metric is of the form $dt^2 + g_t^2$, where g_t is a family of Riemannian metrics on Σ. We can write $b \in \Omega^1(Y)$ as $b = a + \beta dt$, where $a \in \Gamma([0, \epsilon), \Omega^1(\Sigma))$ and $\beta \in \Gamma([0, \epsilon), \Omega^0(\Sigma))$. Let $\check{*}$ denote the Hodge star on Σ with respect to g_0, and let d_Σ be the exterior derivative on Σ.

So with the above notation, we have the following lemma concerning D_{dgc} (where for notational simplicity, we state the result for real-valued forms):

---

[10]More precisely, the range of $r(\ker \mathsf{D})$ and $\mathcal{Z}^+_{\mathsf{B}_0}$ are each given by the range of pseudodifferential projections, and these projections have the same principal symbol. See e.g. [6, 45, 46].

[11]For a general first order differential operator A acting on sections $\Gamma(E)$ over a manifold X, Green's formula for A is the adjunction formula

$$(u, Av)_{L^2(X)} - (A^* u, v)_{L^2(X)} = \int_{\partial X} (r(u), -Jr(v)), \tag{3.65}$$

where A^* is the formal adjoint of A. The map $J : E_{\partial X} \to E_{\partial X}$ is a bundle endomorphism on the boundary and it is determined by A. Hence, (3.66) is an "integration by parts" formula for A. If E is a Hermitian vector bundle, we will always take the real part of (3.65) in order to get a real valued pairing on the boundary.

Lemma 3.8 *Let Y be a 3-manifold with boundary Σ oriented by the outward normal. Then with respect to $(a,\beta,\alpha) \in \Gamma([0,\epsilon), \Omega^1(\Sigma) \oplus \Omega^0(\Sigma) \oplus \Omega^0(\Sigma))$ near the boundary, the div-grad-curl operator can be written as $D_{\mathrm{dgc}} = J_{\mathrm{dgc}}(\frac{d}{dt} + \mathsf{B}_{\mathrm{dgc},t} + \mathsf{C}_{\mathrm{dgc},t})$ as in (3.62), where*[12]

$$J_{\mathrm{dgc}} = \begin{pmatrix} -\check{*} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.67}$$

$$B_{\mathrm{dgc}} = B_{\mathrm{dgc},0} = \begin{pmatrix} 0 & d_\Sigma & \check{*}d_\Sigma \\ d_\Sigma^* & 0 & 0 \\ -\check{*}d_\Sigma & 0 & 0 \end{pmatrix}. \tag{3.68}$$

The positive, negative, and zero eigenspace decompositions for B_{dgc} are given by

$$\mathcal{Z}^\pm_{\mathrm{dgc}} = \mathcal{Z}^\pm_e \oplus \mathcal{Z}^\pm_c \tag{3.69}$$

$$:= \mathrm{span}\left\{ \begin{pmatrix} |\lambda|^{-1} d_\Sigma f_{\lambda^2} \\ \pm f_{\lambda^2} \\ 0 \end{pmatrix} \right\} \oplus \mathrm{span}\left\{ \begin{pmatrix} |\lambda|^{-1} \check{*} d_\Sigma f_{\lambda^2} \\ 0 \\ \pm f_{\lambda^2} \end{pmatrix} \right\} \tag{3.70}$$

$$\mathcal{Z}^0_{\mathrm{dgc}} = H^1(\Sigma;\mathbb{R}) \oplus H^0(\Sigma;\mathbb{R}) \oplus H^0(\Sigma;\mathbb{R}), \tag{3.71}$$

where the f_{λ^2} span the nonzero eigenfunctions of $\Delta = d_\Sigma^* d_\Sigma$ and $\Delta f_{\lambda^2} = \lambda^2 f_{\lambda^2}$.

Let $\Omega^0_\perp(\Sigma) = \{\alpha \in \Omega^0(\Sigma) : \int \alpha = 0\}$ be the span of the nonzero eigenfunctions of Δ. Then for every $s \in \mathbb{R}$, and $1 < p < \infty$, $B^{s,p}\mathcal{Z}^\pm_e$ is the graph of the isomorphism $\pm d_\Sigma \Delta^{-1/2} : B^{s,p}\Omega^0_\perp(\Sigma) \to B^{s,p}\mathrm{im}\, d_\Sigma$. Similarly, the spaces $B^{s,p}\mathcal{Z}^\pm_c$ are graphs of the isomorphisms $\pm\check{*}d_\Sigma\Delta^{-1/2} : B^{s,p}\Omega^0_\perp(\Sigma) \to B^{s,p}\mathrm{im}\,\check{*}d_\Sigma$.

Proof The proof is by direct computation. □

Altogether, we have the following spectral decompositions

$$\widetilde{\mathcal{T}}_\Sigma = \mathcal{Z}^+ \oplus \mathcal{Z}^- \oplus \mathcal{Z}^0, \tag{3.72}$$

$$\Omega^1(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega = \mathcal{Z}^+_{\mathrm{dgc}} \oplus \mathcal{Z}^-_{\mathrm{dgc}} \oplus \mathcal{Z}^0_{\mathrm{dgc}}, \tag{3.73}$$

$$\Gamma(\mathcal{S}_\Sigma) = \mathcal{Z}^+_{\mathcal{S}} \oplus \mathcal{Z}^-_{\mathcal{S}} \oplus \mathcal{Z}^0_{\mathcal{S}}, \tag{3.74}$$

corresponding to the positive, negative, and zero spectral subspaces of B, $\mathsf{B}_{\mathrm{dgc}}$, and $\mathsf{B}_{\mathcal{S}}$, respectively. Since $\mathsf{B} = \mathsf{B}_{\mathrm{dgc}} \oplus \mathsf{B}_{\mathcal{S}}$, we obviously have

$$\mathcal{Z}^\bullet = \mathcal{Z}^\bullet_{\mathrm{dgc}} \oplus \mathcal{Z}^\bullet_{\mathcal{S}}, \qquad \bullet \in \{+,-,0\}. \tag{3.75}$$

In particular, we have

$$\mathcal{Z}^+ = \mathcal{Z}^+_{\mathrm{dgc}} \oplus \mathcal{Z}^+_{\mathcal{S}} \tag{3.76}$$

$$= \mathcal{Z}^+_e \oplus \mathcal{Z}^+_c \oplus \mathcal{Z}^+_{\mathcal{S}}, \tag{3.77}$$

by Lemma 3.8. All the above decompositions hold when we take Besov closures. In light

[12] Note the signs, since t is the *inward* normal coordinate.

of Lemma 3.7, the explicit decomposition (3.77) will be important for us in the analysis to come.

Next, we work out the associated symplectic data for $\widetilde{\mathcal{H}}_0$ on the boundary, following the general picture described previously. Namely, Green's formula (3.66) for the Dirac operator $\widetilde{\mathcal{H}}_0$ induces a symplectic form on the boundary data space $\widetilde{\mathcal{T}}_\Sigma$. Moreover, because $\widetilde{\mathcal{H}}_0$ is a Dirac operator, the endomorphism $-\mathrm{J}_0$ is a compatible complex structure for the symplectic form. Explicitly, the symplectic form is

$$\begin{aligned} \tilde{\omega} : \widetilde{\mathcal{T}}_\Sigma \oplus \widetilde{\mathcal{T}}_\Sigma \to \mathbb{R} \\ \tilde{\omega}((a,\phi,\alpha_1,\alpha_0),(b,\psi,\beta_1,\beta_0)) = \int_\Sigma a \wedge b + \int_\Sigma \mathrm{Re}\,(\phi, \rho(\nu)\psi) - \int_\Sigma (\alpha_1\beta_0 - \alpha_0\beta_1), \end{aligned} \tag{3.78}$$

and the compatible complex structure is

$$\begin{aligned} \tilde{J}_\Sigma : \widetilde{\mathcal{T}}_\Sigma \to \widetilde{\mathcal{T}}_\Sigma \\ (a,\phi,\alpha_1,\alpha_0) \mapsto (-\check{*}a, -\rho(\nu)\phi, -\alpha_0, \alpha_1). \end{aligned} \tag{3.79}$$

Observe that since $\widetilde{\mathcal{H}}_0 = \mathcal{H}_{(B_{\mathrm{ref}},0)} \oplus -(d+d^*)$, the symplectic form and compatible complex structure above are a direct sum of those corresponding to the operators $\mathcal{H}_{(B_{\mathrm{ref}},0)}$ and $(d+d^*)$. In particular, Green's formula for $\mathcal{H}_{(B_{\mathrm{ref}},0)} = *d \oplus D_{B_{\mathrm{ref}}}$ yields the symplectic form

$$\begin{aligned} \omega : \mathcal{T}_\Sigma \oplus \mathcal{T}_\Sigma \to \mathbb{R} \\ \omega((a,\phi),(b,\psi)) = \int_\Sigma a \wedge b + \int_\Sigma \mathrm{Re}\,(\phi, \rho(\nu)\psi) \end{aligned} \tag{3.80}$$

and compatible complex structure

$$\begin{aligned} J_\Sigma : \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma \\ (a,\phi) \mapsto (-\check{*}a, -\rho(\nu)\phi). \end{aligned} \tag{3.81}$$

Since the tangent space to $\mathfrak{C}(\Sigma)$ at any configuration is a copy of $\mathcal{T}_\Sigma$, we see that ω gives us a constant symplectic form on $\mathfrak{C}(\Sigma)$. This symplectic form extends to $\mathfrak{C}^{0,2}(\Sigma)$, the L^2 closure of the configuration space, and since $B^{s,p}(\Sigma) \subseteq B^{0,2}(\Sigma) = L^2(\Sigma)$ for all $s > 0$ and $p \geq 2$, we also get a constant symplectic form on the Besov configuration spaces $\mathfrak{C}^{s,p}(\Sigma)$. From now on, we will always regard $\mathfrak{C}^{s,p}(\Sigma)$ as being endowed with this symplectic structure. Likewise, we always regard $\widetilde{\mathcal{T}}_\Sigma^{s,p}$ as being endowed with the symplectic form (3.78). Indeed, the symplectic forms ω and $\tilde{\omega}$ are the appropriate ones to consider, since they are the symplectic forms induced by the Hessian and augmented Hessian operators, respectively.

Having studied the particular augmented Hessian operator $\widetilde{\mathcal{H}}_0 = \widetilde{\mathcal{H}}_{(B_{\mathrm{ref}},0)}$, we now study general augmented Hessian operators $\widetilde{\mathcal{H}}_{(B,\Psi)}$. Here, $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$ is an arbitrary possibly nonsmooth configuration. Suppose we have a bounded multiplication $B^{s,p}(Y) \times B^{t,q}(Y) \to B^{t-1,q}(Y)$, for some $t \in \mathbb{R}$ and $q \geq 2$. It follows that $\widetilde{\mathcal{H}}_{(B,\Psi)} : \widetilde{\mathcal{T}}^{t,q} \to \widetilde{\mathcal{T}}^{t-1,q}$ and $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{t,q} \to \mathcal{T}^{t-1,q}$ are bounded maps. To keep the topologies clear, we will often use the notation

$$\begin{aligned} \mathcal{H}^{t,q}_{(B,\Psi)} &: \mathcal{T}^{t,q} \to \mathcal{T}^{t-1,q} \\ \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)} &: \widetilde{\mathcal{T}}^{t,q} \to \widetilde{\mathcal{T}}^{t-1,q}, \end{aligned}$$

so that the superscripts on the operators specify the regularity of the domains. The next two lemmas tell us that $\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ and $r(\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)})$ are compact perturbations of $\ker \widetilde{\mathcal{H}}^{t,q}_0$ and $r(\ker \widetilde{\mathcal{H}}^{t,q}_0)$, respectively, for (t,q) in a certain range. We also give a more concrete description of this perturbation using Lemma 18.1.

Lemma 3.9 *Let $s > 3/p$. Let $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$ and suppose $t \in \mathbb{R}$ and $q \geq 2$ are such that we have a bounded multiplication map $B^{s,p}(Y) \times B^{t,q}(Y) \to B^{t'-1,q}(Y)$, where $t' > 1/q$ and $t \leq t' \leq t+1$.*

- *(i) We have that $\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ is commensurate with $\ker \widetilde{\mathcal{H}}^{t,q}_0$ and the restriction map $r : \ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{t-1/q,q}_\Sigma$ is bounded. More precisely, we have the decomposition*

$$\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)} = \{x + \widetilde{T}x : x \in X_0'\} \oplus F, \tag{3.82}$$

where $X_0' \subseteq \ker \widetilde{\mathcal{H}}^{t,q}_0$ has finite codimension, $\widetilde{T} : X_0' \to \widetilde{\mathcal{T}}^{t',q}$, and $F \subseteq \widetilde{\mathcal{T}}^{t',q}$ is a finite dimensional subspace. Moreover, one can choose as a complement for $X_0' \subset \ker \widetilde{\mathcal{H}}^{t,q}_0$ a space that is spanned by smooth elements.

- *(ii) The space $\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ varies continuously[13] with $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$.*

Proof (i) Let $(b,\psi) = (B - B_{\mathrm{ref}}, \Psi)$. The multiplication map $(b,\psi)\# = \widetilde{\mathcal{H}}_{(B,\Psi)} - \widetilde{\mathcal{H}}_0$ given by (3.59) yields a bounded map

$$(b,\psi)\# : \widetilde{\mathcal{T}}^{t',q} \to \widetilde{\mathcal{T}}^{t'-1,q} \tag{3.83}$$

by hypothesis. This map is a compact operator since it is the norm limit of $(b_i,\psi_i)\#$, with (b_i,ψ_i) smooth. Each of the operators $(b_i,\psi_i)\#$ is compact, since it is a bounded operator on $\widetilde{\mathcal{T}}^{t',q}$ and the inclusion $\widetilde{\mathcal{T}}^{t',q} \hookrightarrow \widetilde{\mathcal{T}}^{t'-1,q}$ is compact by Theorem 13.17. Since the space of compact operators is norm closed, this proves (3.83) is compact.

Since $t' > 1/q$, then $\ker \widetilde{\mathcal{H}}^{t',q}_0$ is complemented in $\widetilde{\mathcal{T}}^{t',q}$ by Corollary 15.17. Let $X_1 \subset \widetilde{\mathcal{T}}^{t',q}$ be any such complement. Thus,

$$\widetilde{\mathcal{H}}_0 : X_1 \to \widetilde{\mathcal{T}}^{t'-1,q} \tag{3.84}$$

is injective. It is also surjective by unique continuation, Theorem 17.2. Hence (3.84) is an isomorphism and the map

$$\widetilde{\mathcal{H}}_{(B,\Psi)} : X_1 \to \widetilde{\mathcal{T}}^{t'-1,q}, \tag{3.85}$$

being a compact perturbation of an isomorphism, is Fredholm. This allows us to write the kernel of $\widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ perturbatively as follows.

[13] See Definition 18.9.

Let $x \in \ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$. Then $\widetilde{\mathcal{H}}_0 x = (b,\psi)\#x \in \mathcal{T}^{t'-1,q}$ and we can define

$$x_1 = -\left(\widetilde{\mathcal{H}}_0^{t',q}|_{X_1}\right)^{-1}(b,\psi)\#x \in X_1 \subset \widetilde{\mathcal{T}}^{t',q}.$$

Then if we define $x_0 = x - x_1 \in \widetilde{\mathcal{T}}^{t,q}$, we have

$$\begin{aligned}
\widetilde{\mathcal{H}}_0 x_0 &= \widetilde{\mathcal{H}}_0(x - x_1)\\
&= \left(\widetilde{\mathcal{H}}_{(B,\Psi)} - (b,\psi)\#\right)x - \widetilde{\mathcal{H}}_0 x_1\\
&= -(b,\psi)\#x + (b,\psi)\#x\\
&= 0.
\end{aligned}$$

Hence, $x_0 \in \ker \widetilde{\mathcal{H}}_0^{t,q}$. Thus, we have decomposted $x \in \ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ as $x = x_0 + x_1$, where $x_0 \in \ker \widetilde{\mathcal{H}}_0^{t,q}$ is in the kernel of a smooth operator and $x_1 \in \widetilde{\mathcal{T}}^{t',q}$ is more regular (for $t' > t$). We also have

$$\begin{aligned}
0 &= \widetilde{\mathcal{H}}_{(B,\Psi)} x\\
&= \widetilde{\mathcal{H}}_{(B,\Psi)}(x_1 + x_0)\\
&= \widetilde{\mathcal{H}}_{(B,\Psi)} x_1 + (b,\psi)\#x_0.
\end{aligned} \tag{3.86}$$

By the above, we know that $\widetilde{\mathcal{H}}_{(B,\Psi)} : X_1 \to \widetilde{\mathcal{T}}^{t'-1,q}$ is Fredholm. Thus, from (3.86), we see that there exists a subspace $X_0' \subseteq \ker \widetilde{\mathcal{H}}_0^{t,q}$ of finite codimension such that for all $x_0 \in X_0'$, there exists a solution $x_1 \in X_1$ to (3.86). This solution is unique up to some finite dimensional subspace $F \subset X_1$; in fact F is just the kernel of (3.85). This proves the decomposition (3.82), where the map $\widetilde{T}$ is given by

$$\begin{aligned}
\widetilde{T} : X_0' &\to X_1'\\
x_0 &\mapsto -(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{X_1'})^{-1}(b,\psi)\#x_0,
\end{aligned} \tag{3.87}$$

where X_1' is any complement of $F \subset X_1$. The map $\widetilde{T}$ is compact since the map $(b,\psi)\#$ is compact. The rest of the statement now follows, since the restriction map $r : \ker \widetilde{\mathcal{H}}_0^{t,q} \to \widetilde{\mathcal{T}}_\Sigma^{t-1/q,q}$ is bounded by Theorem 15.14(i), and $r : \widetilde{\mathcal{T}}^{t',q} \to \widetilde{\mathcal{T}}_\Sigma^{t'-1/q,q} \subset \widetilde{\mathcal{T}}_\Sigma^{t-1/q,q}$ is bounded since $t' > 1/q$. Moreover, since smooth elements are dense in $\ker \widetilde{\mathcal{H}}_0^{t,q}$ by Corollary 15.17, any finite dimensional complement for $X_0' \subseteq \ker \widetilde{\mathcal{H}}_0^{t,q}$ can be replaced by a complement that is spanned by smooth elements if necessary.

(ii) Let $(B_0,\Psi_0) \in \mathfrak{C}^{s,p}(Y)$. By Definition 18.9, we have to show that $\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ is a graph over $\ker \widetilde{\mathcal{H}}^{t,q}_{(B_0,\Psi_0)}$ for (B,Ψ) close to (B_0,Ψ_0). We do the same thing as in (i). Let X_2 be any complement of $\ker \widetilde{\mathcal{H}}^{t',q}_{(B_0,\Psi_0)}$ in $\widetilde{\mathcal{T}}^{t',q}$, which exists since $\ker \widetilde{\mathcal{H}}^{t',q}_{(B_0,\Psi_0)}$ is commensurate with $\ker \widetilde{\mathcal{H}}_0^{t',q}$ by (i), and the latter space is complemented. Then $\widetilde{\mathcal{H}}_{(B_0,\Psi_0)} : X_2 \to \widetilde{\mathcal{T}}^{t'-1,q}$ is an isomorphism. For (B,Ψ) sufficiently close to (B_0,Ψ_0), the map $\widetilde{\mathcal{H}}_{(B,\Psi)} : X_2 \to \widetilde{\mathcal{T}}^{t'-1,q}$ is injective, hence surjective (the index is invariant under compact perturbations), and therefore an isomorphism. Then from the above analysis,

$$\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)} = \{x + \widetilde{T}_{(B,\Psi)}x : x \in \ker \widetilde{\mathcal{H}}^{t,q}_{(B_0,\Psi_0)}\}, \tag{3.88}$$

where

$$\widetilde{T}_{(B,\Psi)} : \ker \widetilde{\mathcal{H}}^{t,q}_{(B_0,\Psi_0)} \to X_2 \tag{3.89}$$

$$x \mapsto -(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{X_2})^{-1}(b,\psi)\#x \tag{3.90}$$

and $(b,\psi) = (B - B_0, \Psi - \Psi_0)$. The map $\widetilde{T}_{(B,\Psi)}$ varies continuously with $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$ near (B_0,Ψ_0). $\square$

Remark 3.10 In applications of the above lemma, instead of (t,q) satisfying the very general hypothesis

$$\left.\begin{array}{ll}\text{(i)} & t \in \mathbb{R},\ q \geq 2, \\ \text{(ii)} & \text{the multiplication } B^{s,p}(Y) \times B^{t,q}(Y) \to B^{t'-1,q}(Y) \text{ is bounded, where } t' > 1/q \\ & \text{and } t \leq t' \leq t+1,\end{array}\right\} \tag{3.91}$$

we will primarily only need the cases

$$(t,q) \in \{(s+1,p),\ (s,p),\ (1/2,2)\}, \tag{3.92}$$

with corresponding values

$$(t',q) \in \{(s+1,p),\ (s+1,p),\ (1/2+\epsilon,2)\}, \qquad \epsilon > 0. \tag{3.93}$$

The last case of (3.92) arises because we want to consider the space of boundary values in the L^2 topology, i.e., the spaces $\mathcal{T}^{0,2}_\Sigma$ and $\widetilde{\mathcal{T}}^{0,2}_\Sigma$. In this particular case, the above lemma allows us to conclude that for $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, we still get bounded restriction maps $r : \ker \widetilde{\mathcal{H}}^{1/2,2}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{0,2}_\Sigma$, just like in the case where (B,Ψ) is smooth via Theorem 15.14. The boundedness of this map will be important when we perform symplectic reduction on Banach spaces in the proof of Theorem 3.13. The case $(t,q) = (s+1,p)$ will be important for Proposition 3.20 and its applications in Section 4. In what follows, we will consider the operators $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ but they equally well apply to $\widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ in light of the analysis in Lemma 3.9, for t,q satisfying (3.91).

Lemma 3.11 *Let $s > 3/p$. For any $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, we have the following:*

(i) The Cauchy data space $r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ is a Lagrangian subspace of $\widetilde{\mathcal{T}}^{s-1/p,p}_\Sigma$ commensurate with $B^{s-1/p,p}\mathcal{Z}^+$ and it varies continuously with (B,Ψ).

(ii) We have a direct sum decomposition $\widetilde{\mathcal{T}}^{s-1/p,p}_\Sigma = r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) \oplus \tilde{J}_\Sigma r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$.

Proof (i) For any $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, the space $r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ is isotropic since $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ is formally self-adjoint. Since $s > 3/p$, then $(B,\Psi) \in L^\infty(Y)$ and we can apply the unique continuation theorem, Theorem 17.1, which implies that $r : \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ is injective. In fact, it is an isomorphism onto its image, since this is true for $r : \ker \widetilde{\mathcal{H}}^{s,p}_0 \to \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ (by Theorem 15.14(i) and unique continuation applied to the smooth operator $\widetilde{\mathcal{H}}^{s,p}_0$) and $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ is a compact perturbation of $\ker \widetilde{\mathcal{H}}^{s,p}_0$ by Lemma 3.9. Hence, we get that $\widetilde{L}^{s-1/p,p}_{(B,\Psi)} := r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ varies continuously with (B,Ψ), since $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ varies continuously by Lemma 3.9 and $r : \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ is an isomorphism onto its image. For (B,Ψ) smooth, we know that $\widetilde{L}^{s-1/p,p}_{(B,\Psi)} \subset \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ is a Lagrangian subspace by Proposition 15.18. By continuity then, $\widetilde{L}^{s-1/p,p}_{(B,\Psi)}$ is a Lagrangian for all $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. Moreover, all the $\widetilde{L}^{s-1/p,p}_{(B,\Psi)}$ are commensurate with one another, in particular, with $r(\ker \widetilde{\mathcal{H}}^{s-1/p,p}_0)$, and this latter space is commensurate with $B^{s-1/p,p}\mathcal{Z}^+$ by Lemma 3.7.

(ii) When (B,Ψ) is smooth, this follows from Proposition 15.18. Now we use the continuity of the Lagrangians with respect to $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$ for the general case. □

We want to apply the previous results concerning the augmented Hessian $\widetilde{\mathcal{H}}_{(B,\Psi)}$ to deduce properties about the Hessian $\mathcal{H}_{(B,\Psi)}$. To place these results in a context similar to the pseudodifferential picture in Section 15.3, let us recall some more basic properties concerning the smooth operator $\widetilde{\mathcal{H}}_0$. By Theorem 15.14, the operator $\widetilde{\mathcal{H}}_0$, by virtue of it being a smooth elliptic operator, has a Calderon projection $\widetilde{P}^+_0$ and a Poisson operator $\widetilde{P}_0$. These operators satisfy the following properties. The map $\widetilde{P}^+_0$ is a projection of the boundary data $\widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ onto $r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$, the boundary values of $\ker \widetilde{\mathcal{H}}^{s,p}_0$, and the map $\widetilde{P}_0$ is a map from the boundary data $\widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ into $\ker(\widetilde{\mathcal{H}}^{s,p}_0) \subset \widetilde{\mathcal{T}}^{s,p}$. Moreover, the maps $r : \ker \widetilde{\mathcal{H}}^{s,p}_0 \to r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$ and $\widetilde{P}_0 : r(\ker \widetilde{\mathcal{H}}^{s,p}_0) \to \ker \widetilde{\mathcal{H}}^{s,p}_0$ are inverse to one another, and $r\widetilde{P}_0 = \widetilde{P}^+_0$. This implies that the map $\tilde{\pi}_0 := \widetilde{P}_0 r : \widetilde{\mathcal{T}}^{s,p} \to \ker(\widetilde{\mathcal{H}}^{s,p}_0)$ is a projection. We also have that $\mathrm{im}\, \widetilde{P}^+_0 = r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$ is a Lagrangian subspace of $B^{s-1/p,p}\widetilde{\mathcal{T}}_{\Sigma}$ by Proposition 15.18.

For a general nonsmooth $(B,\Psi) \in \mathfrak{C}^{s,p}$, we have $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ is a compact perturbation of the smooth elliptic operator $\widetilde{\mathcal{H}}^{s,p}_0$. The previous lemmas imply that $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ and $r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ are compact perturbations of $\ker \widetilde{\mathcal{H}}^{s,p}_0$ and $r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$, respectively, and moreover, we still have unique continuation, i.e., $r : \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ is an isomorphism onto its image. It follows that there exists a Calderon projection $\widetilde{P}^+_{(B,\Psi)}$ and Poisson operator $\widetilde{P}_{(B,\Psi)}$ for $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ as well, which satisfy the same corresponding properties (see Lemma 18.1). We also have a projection $\tilde{\pi}_{(B,\Psi)} := \widetilde{P}_{(B,\Psi)} r : \widetilde{\mathcal{T}}^{s,p} \to \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$. We summarize this in the following lemma and diagram:

Lemma 3.12 *Let $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. Then there exists a Calderon projection $\widetilde{P}^+_{(B,\Psi)} : \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma} \to r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ and a Poisson operator $\widetilde{P}_{(B,\Psi)} : \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma} \to \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$. The maps $r : \ker \widetilde{\mathcal{H}}_{(B,\Psi)} \to r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ and $\widetilde{P}_{(B,\Psi)} : r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) \to \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ are inverse to one another, and $r\widetilde{P}_{(B,\Psi)} = \widetilde{P}^+_{(B,\Psi)}$.*

$$\begin{array}{ccccc} \ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)} & \xleftarrow{\tilde{\pi}_{(B,\Psi)}} & \widetilde{\mathcal{T}}^{s,p} & \xrightarrow{\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}} & \widetilde{\mathcal{T}}^{s-1,p} \\ r \downarrow \uparrow \widetilde{P}_{(B,\Psi)} & & \downarrow r & & \\ r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) & \xleftarrow[\widetilde{P}^{+}_{(B,\Psi)}]{} & \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma} & & \end{array} \tag{3.94}$$

In studying the Hessian $\mathcal{H}^{s,p}_{(B,\Psi)}$ we want to establish similar results as in Lemmas 3.11 and 3.12. These results are summarized in the main theorem of this section.

Theorem 3.13 *Let $s > \max(3/p, 1/2)$ and let $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$. Suppose $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is surjective.*[14] *Then we have the following:*

(i) The space $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ is a Lagrangian subspace of $\mathcal{T}^{s-1/p,p}_\Sigma$ commensurate with $B^{s-1/p,p}(\operatorname{im} d \oplus \mathcal{Z}^+_{\mathcal{S}})$. Moreover, we have the direct sum decomposition

$$\mathcal{T}^{s-1/p,p}_\Sigma = r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) \oplus J_\Sigma r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}). \tag{3.95}$$

(ii) Define

$$P^+_{(B,\Psi)} : \mathcal{T}^{s-1/p,p}_\Sigma \to r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) \tag{3.96}$$

to be the projection onto $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ through $J_\Sigma r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ as given by (3.95). Let $\pi^+ : \mathcal{T}_\Sigma \to \operatorname{im} d \oplus \mathcal{Z}^+_{\mathcal{S}}$ denote the orthogonal projection onto $\operatorname{im} d \oplus \mathcal{Z}^+_{\mathcal{S}}$ through the complementary space $\ker d^* \oplus (\mathcal{Z}^-_{\mathcal{S}} \oplus \mathcal{Z}^0_{\mathcal{S}})$. Then π^+, being a pseudodifferential projection, extends to a bounded map on $\mathcal{T}^{s-1/p,p}_\Sigma$, and it differs from the projection $P^+_{(B_0,\Psi_0)}$ by an operator

$$(P^+_{(B,\Psi)} - \pi^+) : \mathcal{T}^{s-1/p,p}_\Sigma \to \mathcal{T}^{s-1/p+1,p}_\Sigma. \tag{3.97}$$

which smooths by one derivative.

(iii) There exists a unique operator

$$P_{(B,\Psi)} : \mathcal{T}^{s-1/p,p}_\Sigma \to \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \tag{3.98}$$

that satisfies $r_\Sigma P_{(B,\Psi)} = P^+_{(B,\Psi)}$. The maps $r_\Sigma : \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \to r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ and $P_{(B,\Psi)} : r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) \to \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})$ are inverse to one another.

Furthermore, let $(B(t),\Psi(t))$ be a continuous (smooth) path in $\mathfrak{M}^{s,p}(Y)$ such that $\mathcal{H}_{(B(t),\Psi(t))} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B(t),\Psi(t))}$ is surjective for all t.

[14] This holds under the assumption (4.1). See Lemma 4.1.

(iv) Then $\ker \mathcal{H}^{s,p}_{(B(t),\Psi(t))}$ *and* $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B(t),\Psi(t))})$ *are continuously (smoothly) varying families of subspaces*[15]*. Consequently, the corresponding operators* $P^+_{(B_t,\Psi_t)}$ *and* $P_{(B_t,\Psi_t)}$ *vary continuously (smoothly) in the operator norm topologies.*

Keeping $\mathfrak{M}^{s,p}(Y)$ *fixed, the statements in (i), (iii), and (iv) remain true if we replace the* $B^{s,p}(Y)$ *and* $B^{s-1/p,p}(\Sigma)$ *topologies on all vector spaces with the* $B^{t,q}(Y)$ *and* $B^{t-1/q,q}(\Sigma)$ *topologies, respectively, where* t, q *satisfy (3.92) or more generally (3.91). If we do the same for (ii), everything also holds except that the map* (3.97) *smooths by* $t' - t$ *derivatives.*

The theorem implies we have the following corresponding diagram for the Hessian $\mathcal{H}_{(B,\Psi)}$:

$$\begin{array}{ccccc} \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) & \xleftarrow{\pi_{(B,\Psi)}} & \mathcal{T}^{s,p} & \xrightarrow{\mathcal{H}^{s,p}_{(B,\Psi)}} & \mathcal{T}^{s-1,p} \\ r_\Sigma \downarrow \uparrow P_{(B,\Psi)} & & \downarrow r_\Sigma & & \\ r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) & \xleftarrow[P^+_{(B,\Psi)}]{} & \mathcal{T}^{s-1/p,p}_\Sigma & & \end{array} \tag{3.99}$$

Here, $\pi_{(B,\Psi)} := P_{(B,\Psi)} r_\Sigma$ is a projection of $\mathcal{T}^{s,p}$ onto $\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})$.

Definition 3.14 By abuse of language, we call the operators $P^+_{(B,\Psi)}$ and $P_{(B,\Psi)}$ defined in Theorem 3.13 the *Calderon projection* and *Poisson operator* associated to $\mathcal{H}^{s,p}_{(B,\Psi)}$, respectively (even though $\mathcal{H}^{s,p}_{(B,\Psi)}$ is not an elliptic operator), due to their formal resemblance to Calderon and Poisson operators for elliptic operators (as seen in the diagrams (3.94) and (3.99)).

Note that the Calderon projection $P^+_{(B,\Psi)}$ and Poisson operator $P_{(B,\Psi)}$ we define above are unique, since we specified their kernels. In the general situation of an elliptic operator (such as $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ above) one usually only specifies the range of the Calderon projection, in which case, the projection is not unique (see also Remark 15.16). Our particular choice of kernel for $P^+_{(B,\Psi)}$ is made so that $P^+_{(B,\Psi)}$ is nearly pseudodifferential, in the sense of the smoothing property (3.97). This property will be used in Part III, where analytic properties of the tangent spaces to the Lagrangian $\mathcal{L}^{s-1/p,p}$ and the projections onto them become crucial.

Remark 3.15 The continuous (smooth) dependence of $P^+_{(B,\Psi)}$ and $P_{(B,\Psi)}$ in Theorem 3.13(iii) with respect to (B, Ψ), as well as all other continuous dependence statements appearing in the rest of Part I, will only attain their true significance in Part III. There, we will consider paths of configurations, and so naturally, we will have to consider time-varying objects. For brevity, we will only make statements regarding continuous dependence from now on, though they can all be adapted to smooth dependence with no change in argument.

Proving Theorem 3.13 is essentially deducing diagram (3.99) from diagram (3.94). Let us first make sense of the hypotheses of the theorem. From Lemma 3.4, in order for $\mathcal{K}^{s-1,p}_{(B,\Psi)}$

[15] See Definition 18.9.

to be well-defined when $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, we need $s > \max(1-s, 3/p)$, which means we need $s > \max(3/p, 1/2)$. This explains the first hypothesis. Next, observe that for $(B,\Psi) \in \mathcal{M}(Y)$ a smooth monopole, we have

$$\mathcal{J}_{(B,\Psi)} \subseteq \ker \mathcal{H}_{(B,\Psi)}, \tag{3.100}$$

$$\operatorname{im} \mathcal{H}_{(B,\Psi)} \subseteq \mathcal{K}_{(B,\Psi)}. \tag{3.101}$$

One can verify this directly by a computation or reason as follows. As previously discussed, the Seiberg-Witten map (2.2) is gauge-equivariant and hence its set of zeros is gauge-invariant. Thus, the derivative of SW_3 along the gauge-orbit of a monopole vanishes. This is precisely (3.100). For (3.101), observe that the range of $\mathcal{H}_{(B,\Psi)}$ annihilates $\mathcal{J}_{(B,\Psi),t}$ by (3.100) and since $\mathcal{H}_{(B,\Psi)}$ is formally self-adjoint. From the orthogonal decomposition $\mathcal{T} = \mathcal{J}_{(B,\Psi),t} \oplus \mathcal{K}_{(B,\Psi)}$, we conclude that $\operatorname{im} \mathcal{H}_{(B,\Psi)} \subseteq \mathcal{K}_{(B,\Psi)}$. We want to establish similar properties on Besov spaces. Namely, we want

$$\mathcal{J}^{s,p}_{(B,\Psi)} \subseteq \ker \mathcal{H}^{s,p}_{(B,\Psi)}, \tag{3.102}$$

$$\operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)} \subseteq \mathcal{K}^{s-1,p}_{(B,\Psi)}. \tag{3.103}$$

However, this follows formally from (3.100) and (3.101) as long as we can establish on Besov spaces the appropriate mapping properties of the differentiation and multiplication involved in verifying (3.102) and (3.103) directly. Thus, (3.102) holds because the map $\widetilde{\mathcal{H}}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{T}^{s-1,p}$ is bounded when $s > 3/p$. Likewise, (3.101) holds because $\mathbf{d}^*_{(B,\Psi)} : \mathcal{T}^{s-1,p} \to B^{s-2,p}\Omega^0(Y; i\mathbb{R})$ is bounded when $s > \max(3/p, 1/2)$. In drawing these conclusions, as done everywhere else in Part I, we make essential use of Corollary 13.14 and Theorem 13.18.

Thus, from (3.103), we see that the hypotheses of Theorem 3.13 make sense. In fact, for $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$, we have the following result concerning the range of $\mathcal{H}^{s,p}_{(B,\Psi)}$:

Lemma 3.16 *Let $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$. Then $\operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)} \subseteq \mathcal{K}^{s-1,p}_{(B,\Psi)}$ and $\mathcal{H}^{s,p}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ has closed range and finite dimensional cokernel.*

Proof It remains to prove the final statement. Pick any elliptic boundary condition for the operator $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ such that one of the boundary conditions for $(b,\psi,\alpha) \in \widetilde{\mathcal{T}}^{s,p}$ is $\alpha|_\Sigma = 0$. Such a boundary condition is possible, since the subspace $B^{s-1/p,p}(\mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus 0)$ of $\widetilde{\mathcal{T}}^{s-1/p,p}_\Sigma$ with vanishing $0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma)$ component contains subspaces Fredholm[16] with $r(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ by Lemma 3.7, Lemma 3.8, and (3.77). For such a boundary condition, observe that $\operatorname{im}(\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) \cap \mathcal{K}^{s-1,p}_{(B,\Psi)} \subseteq \operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)}$. This is because if $d\alpha \in \mathcal{K}^{s-1,p}_{(B,\Psi)}$ with $\alpha|_\Sigma = 0$, then $d\alpha = 0$. Since we chose elliptic boundary conditions for $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$, this means $\operatorname{im} \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)} \subseteq \widetilde{\mathcal{T}}^{s-1,p}_{(B,\Psi)}$ is closed and has finite codimension, which implies $\operatorname{im}(\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) \cap \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is also closed and has finite codimension in $\mathcal{K}^{s-1,p}_{(B,\Psi)}$. Hence, the same is true for $\operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)}$. □

Next, we relate the kernel of $\mathcal{H}^{s,p}_{(B,\Psi)}$ to the kernel of $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ along with their respective boundary values.

[16] See Definition 18.4.

Lemma 3.17 *Let $s > \max(3/p, 1/2)$ and $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$.*

(i) We have a decomposition

$$\ker \mathcal{H}^{s,p}_{(B,\Psi)} = \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \oplus \mathcal{J}^{s,p}_{(B,\Psi),t} \tag{3.104}$$

$$\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)} = \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \oplus \Gamma^{s,p}_0, \tag{3.105}$$

where $\Gamma^{s,p}_0 \subseteq \widetilde{\mathcal{T}}^{s,p}$ is the graph of a map $\Theta_0 : \ker\Delta \dashrightarrow \mathcal{T}^{s,p}$, where Δ is the Laplacian on $B^{s,p}\Omega^0(Y; i\mathbb{R})$, and the domain of Θ_0 has finite codimension.

(ii) We have

$$r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) = r_\Sigma(\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})) \tag{3.106}$$

$$r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) = r(\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})) \oplus \check{\Gamma}^{s-1/p,p}_0, \tag{3.107}$$

where $\check{\Gamma}^{s-1/p,p}_0 = r(\Gamma^{s,p}_0)$ is the graph of a map $\check{\Theta}_0 : 0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R}) \dashrightarrow \mathcal{T}^{s-1/p,p}_\Sigma$ and the domain of $\check{\Theta}_0$ has finite codimension.

(iii) We have $r(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ is commensurate with $B^{s-1/p,p}(\mathcal{Z}^+_e \oplus \mathcal{Z}^+_{\mathcal{S}})$ and $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ is commensurate with $B^{s-1/p,p}(\operatorname{im} d \oplus \mathcal{Z}^+_{\mathcal{S}})$.

Proof (i) The first decomposition (3.104) follows from (3.38) and $\mathcal{J}^{s,p}_{(B,\Psi),t} \subset \ker \mathcal{H}^{s,p}_{(B,\Psi)}$. For (3.105), observe that $\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}_{(B,\Psi)}}) = \ker \widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{T}^{s,p}}$. Thus, the elements of $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ that do not lie in $\ker \widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{T}^{s,p}}$ have nonzero $B^{s,p}\Omega^0(Y; i\mathbb{R})$ component. To find them, we need to solve the equation

$$\mathcal{H}_{(B,\Psi)}(b,\psi) - d\alpha = 0, \tag{3.108}$$

with α nonzero. Since $\operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)} \subseteq \mathcal{K}^{s-1,p}_{(B,\Psi)}$ by the previous lemma, we need $d\alpha \in \mathcal{K}^{s-1,p}_{(B,\Psi)}$, whence $\alpha \in \ker\Delta$. Since $\operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)}$ has finite codimension in $\mathcal{K}^{s-1,p}_{(B,\Psi)}$ by Lemma 3.16, then (3.108) has a solution (b,ψ) for all α in some subspace of $\ker\Delta$ of finite codimension. The (b,ψ) is unique up to an element of $\ker \mathcal{H}^{s,p}_{(B,\Psi)}$. Thus, picking a complement[17] of $\ker \mathcal{H}^{s,p}_{(B,\Psi)}$ in $\mathcal{T}^{s,p}_{(B,\Psi)}$ specifies for us a map $\Theta_0 : \ker\Delta \dashrightarrow \mathcal{T}^{s,p}$ whose graph $\Gamma^{s,p}_0$ is a complementary subspace of $\ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{T}^{s,p}_{(B,\Psi)}})$ in $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$, and which parametrizes solutions to (3.108).

(ii) This follows from applying r_Σ and r to (i). The graph property of $\check{\Gamma}^{s-1/p,p}_0$ comes from noting that any element of $\Gamma^{s,p}_0$ is uniquely determined by the $0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R})$ component of its image under r. This follows from considering the homogeneous Dirichlet problem for Δ, namely

$$\begin{aligned} \Delta\alpha &= 0 \\ \alpha|_\Sigma &= \beta. \end{aligned}$$

This problem has a unique solution for every β.

[17]The reasoning used in the proof of Lemma 3.16 shows that $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ has a right parametrix. This implies that $\ker \mathcal{H}^{s,p}_{(B,\Psi)} \subset \mathcal{T}^{s,p}$ is complemented.

(iii) By Lemma 3.7, we have $r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ is commensurate with $B^{s-1/p,p}\mathcal{Z}^+$. Let

$$\pi_0 : \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma} \to 0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R})$$

denote the coordinate projection onto the last 0-form factor in $\widetilde{\mathcal{T}}^{s-1/p}_{\Sigma}$. By Lemma 3.8 and (ii), we have

$$\pi_0 : B^{s-1/p,p}\mathcal{Z}^+_c \to 0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R})$$
$$\pi_0 : B^{s-1/p,p}\check{\Gamma}_0 \to 0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R})$$

are Fredholm. We now apply Lemma 18.6 with $X = \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ and complementary subspaces

$$X_1 = \mathcal{T}^{s-1/p,p}_{\Sigma} \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R}) \oplus 0,$$
$$X_0 = 0 \oplus 0 \oplus B^{s-1/p,p}\Omega^0(\Sigma; i\mathbb{R}).$$

Let $U = r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ and $V = B^{s-1/p,p}\mathcal{Z}^+$ in the lemma. Then from that lemma and (ii), we conclude that $r(\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})) = U \cap X_1$ is commensurate with $B^{s-1/p,p}(\mathcal{Z}^+_e \oplus \mathcal{Z}^+_{\mathcal{S}}) = V \cap X_1$. This proves the first part of (iii). For the second part, consider the coordinate projection of X_1 onto $\mathcal{T}^{s-1/p,p}_{\Sigma}$. This restricts to an isomorphism of $V \cap X_1$ onto its image $B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$, by Lemma 3.8. It follows that this projection maps $U \cap X_1$ onto a space commensurate with $V \cap X_1$, and this space is precisely $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$. □

Corollary 3.18 *Let $(B, \Psi) \in \mathfrak{M}^{s,p}(Y)$ and suppose $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is surjective. Then*

(i) the maps $\Theta^{s,p}_0$ and $\check{\Theta}^{s-1/p,p}_0$ are defined everywhere;

(ii) $r_\Sigma : \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \to \mathcal{T}^{s-1/p,p}_{\Sigma}$ is an isomorphism onto its image.

Proof (i) This follows from the constructions of Θ_0 and $\check{\Theta}_0$ in the previous lemma.

(ii) By unique continuation, the map $r : \ker(\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}) \to \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ is an isomorphism onto its range. By restriction, it follows that

$$r : \ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \to \widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma} \tag{3.109}$$

is injective. To prove (ii), it suffices to show that

$$r_\Sigma : \ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \to \mathcal{T}^{s-1/p,p}_{\Sigma} \tag{3.110}$$

is injective, since $\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{C}^{s,p}} = \mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}$. So suppose (3.110) is not injective. Since (3.109) is injective, this means there is an element of the form $((0,0), \alpha, 0) \in r(\ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}))$ with $\alpha \in B^{s-1/p,p}\Omega^0(\Sigma)$ nonzero. On the other hand, $r(\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)})$ is a Lagrangian subspace of $\widetilde{\mathcal{T}}^{s-1/p,p}_{\Sigma}$ by Lemma 3.11. This contradicts (i), since if $\check{\Theta}^{s-1/p,p}_0$ is defined everywhere, then $(0, 0, \alpha, 0)$ cannot symplectically annihilate $\check{\Gamma}^{s-1/p,p}_0$. Indeed, the spaces $0 \oplus \Omega^0(\Sigma) \oplus 0$ and

$0 \oplus 0 \oplus \Omega^0(\Sigma)$ are symplectic conjugates with respect to the symplectic form (3.78). □

Proof of Theorem 3.13: (i) We will apply the method of symplectic reduction, via Theorem 19.1 and Corollary 19.2. By Lemma 3.9, we may consider the operators $\widetilde{\mathcal{H}}^{1/2,2}_{(B,\Psi)}$ and $\mathcal{H}^{1/2,2}_{(B,\Psi)}$, their kernels, and the restrictions of these latter spaces to the boundary. Indeed, let us verify the hypotheses of Lemma 3.9. Since $p \geq 2$, we have the embedding $B^{s,p}(\Sigma) \subseteq B^{s-\epsilon,2}(\Sigma)$ for any $\epsilon > 0$ by Theorem 13.17. Choose ϵ small enough so that $s - \epsilon > 1/2 + \epsilon$. Then $(t,q) = (1/2,2)$ and $t' = \frac{1}{2} + \epsilon$ satisfies the hypotheses of Lemma 3.9 since we have $B^{s-\epsilon,2}(Y) \times B^{1/2,2}(Y) \to B^{t'-1,2}(Y)$.

Let $U = L^2(\mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus 0)$. It is a coisotropic subspace of the strongly symplectic Hilbert space $L^2\widetilde{\mathcal{T}}_\Sigma = \widetilde{\mathcal{T}}^{0,2}_\Sigma$. If we apply Theorem 19.1 to the Lagrangian $L = r(\ker \widetilde{\mathcal{H}}^{1/2,2}_{(B,\Psi)})$, the symplectic reduction of L with respect to U is precisely $r_\Sigma(\ker \widetilde{\mathcal{H}}^{1/2,2}_{(B,\Psi)})$ by Lemma 3.17(ii). It follows that $r_\Sigma(\ker \mathcal{H}^{1/2,2}_{(B,\Psi)})$ is a Lagrangian inside $U \cap \tilde{J}_\Sigma U = L^2\mathcal{T}_\Sigma$. We would like to make the corresponding statement in the Besov topologies. By Lemma 3.17(iii), we know that $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ is commensurate with $B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$. On the other hand, we have that $B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$ and $J_\Sigma B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$ are Fredholm in $\mathcal{T}^{s-1/p,p}_\Sigma$. Indeed, the Hodge decomposition implies $\mathrm{im}\, d$ and $\mathrm{im} * d$ are Fredholm in $B^{s-1/p,p}\Omega^1(\Sigma; i\mathbb{R})$, and since $\rho(\nu)$ interchanges the positive and negative eigenspaces $\mathcal{Z}^+_{\mathcal{S}}$ and $\mathcal{Z}^-_{\mathcal{S}}$ of the tangential boundary operator $\mathsf{B}_{\mathcal{S}}$ associated to the spinor Dirac operator $D_{B_{\mathrm{ref}}}$, we have that the $B^{s-1/p,p}(\Sigma)$ closures of $\mathcal{Z}^+_{\mathcal{S}}$ and $\rho(\nu)\mathcal{Z}^+_{\mathcal{S}} = \mathcal{Z}^-_{\mathcal{S}}$ are Fredholm in $B^{s-1/p,p}\Gamma(\mathcal{S})$. That these decompositions are Fredholm in Besov topologies follows from the fact these spaces are given by the range of pseudodifferential projections whose principal symbols are complementary projections, and pseudodifferential operators are bounded on Besov spaces. We now apply Corollary 19.2, with $X = \mathcal{T}^{0,2}_\Sigma$ and $Y = \mathcal{T}^{s-1/p,p}_\Sigma$, to conclude that $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ is a Lagrangian subspace of $B^{s-1/p,p}\mathcal{T}_\Sigma$.

(ii) By Lemma 3.11 and (i), $r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$ is commensurate with $B^{s-1/p,p}\mathcal{Z}^+$ and $r_\Sigma(\ker \mathcal{H}^{s,p}_0)$ is commensurate with $B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$, respectively. Since $\widetilde{\mathcal{H}}_0$ is smooth, then we can even say more: there exist pseudodifferential projections onto $r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$ and $B^{s-1/p,p}\mathcal{Z}^+$ that have the same principal symbol, which means that their difference is a pseudodifferential operator of order -1. It follows that the projection of $r(\ker \widetilde{\mathcal{H}}^{s,p}_0)$ onto any complement[18] of $B^{s-1/p,p}\mathcal{Z}^+$ is smoothing of order one. Consequently, letting $U^{s-1/p,p} = B^{s-1/p,p}(\mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus 0)$, then the projection of $r(\ker \widetilde{\mathcal{H}}^{s,p}_0) \cap U$ onto any complement of $B^{s-1/p,p}\mathcal{Z}^+ \cap U$ is smoothing of order one. (Here, we use the fact that $U^{s-1/p,p} + B^{s-1/p,p}\mathcal{Z}^+$ has finite codimension in $\widetilde{\mathcal{T}}^{s-1/p,p}_\Sigma$.) Applying symplectic reduction with respect to $U^{s-1/p,p}$, it follows that the projection of $r_\Sigma(\ker \mathcal{H}^{s,p}_0)$ onto any complement of $B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$ is smoothing of order one.

For a nonsmooth configuration (B,Ψ), we also want to show that the projection of $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ onto any complement of $B^{s-1/p,p}(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$ is smoothing of order one. For then this will imply the corresponding property with respect to the pair of spaces

[18] More precisely, in what follows, when we speak of some unspecified complementary subspace, we mean one defined by a pseudodifferential projection. This is convenient because pseudodifferential operators preserve regularity, i.e., they map $B^{t,q}(\Sigma)$ to itself for all $t, q \in \mathbb{R}$, and so we never lose any smoothness once we have gained it.

$J_\Sigma(r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}))$ and $B^{s-1/p,p}J_\Sigma(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}})$, the latter being of finite codimension in $B^{s-1/p,p}(\ker d^* \oplus (\mathcal{Z}^+_{\mathcal{S}} \oplus \mathcal{Z}^0_{\mathcal{S}}))$. We can then apply Lemma 18.7(ii) while noting Remark 18.8, to conclude that the projection $P^+_{(B_0,\Psi_0)}$ differs from π^+ by a operator that is smoothing of order one.

Thus, by our first step, it suffices to show that the projection of $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ onto any complement of $r_\Sigma(\ker \mathcal{H}^{s,p}_0)$ smooths by one derivative. This follows however from Lemma 3.9. Indeed, we can take $(t,q) = (s,p)$ and $t' = s+1$ in Lemma 3.9, and since there exists a projection of $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ onto a complement of $\ker \widetilde{\mathcal{H}}^{s,p}_0$ that smooths by one derivative, the corresponding statement is true for the spaces $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$ and $r_\Sigma(\ker \mathcal{H}^{s,p}_0)$. Here, it is important that all finite dimensional errors involved are spanned by elements that are smoother by one derivative (so that the finite rank projection onto the space spanned by these elements smooths by one derivative), which is guaranteed by Lemma 3.9. From these properties, one can now apply Lemma 18.7(ii), with

$$\begin{aligned}
X &= \mathcal{T}^{s-1/p,p}_\Sigma \\
U_0 &= B^{s-1/p,p}\big(\mathrm{im}\, d \oplus \mathcal{Z}^+_{\mathcal{S}}\big) \\
U_1 &= B^{s-1/p,p}\big(\ker d^* \oplus (\mathcal{Z}^-_{\mathcal{S}} \oplus \mathcal{Z}^0_{\mathcal{S}})\big) \\
V_1 &= r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) \\
V_2 &= J_\Sigma(r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})).
\end{aligned}$$

In our case, we know that $X = U_0 \oplus U_1 = V_0 \oplus V_1$, and that the U_i and V_i are commensurate, $i = 0,1$, where the compact error is smoothing of order one. Thus, by Remark 18.8, $P^+_{(B_0,\Psi_0)} = \pi_{V_0,V_1}$ and $\pi^+ = \pi_{U_0,U_1}$ differ by an operator that smooths of order one.

(iii) Let

$$\pi_{SR} : r(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) \to r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$$

be the symplectic reduction as in (i), i.e., the map π_{SR} is the map which projects $r(\ker \mathcal{H}^{s,p}_{(B,\Psi)}) \subset \widetilde{\mathcal{T}}^{s-1/p,p}_\Sigma$ onto $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B,\Psi)})$, induced by the projection $\widetilde{\mathcal{T}}^{s-1/p,p}_\Sigma \to \mathcal{T}^{s-1/p,p}_\Sigma$ onto the first factor. This map is an isomorphism by Corollary 3.18(ii). Hence, π^{-1}_{SR} exists and is bounded. Define

$$P_{(B,\Psi)} = \widetilde{P}_{(B,\Psi)}(\pi_{SR})^{-1}P^+_{(B,\Psi)}, \tag{3.111}$$

where $\widetilde{P}_{(B,\Psi)}$ is the Poisson operator of $\widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$. By construction, $P^+_{(B,\Psi)} : \mathcal{T}^{s-1/p,p}_\Sigma \to r_\Sigma(\ker \mathcal{H}^{s,p})$ and $\tilde{P}_{(B,\Psi)}(\pi_{SR})^{-1} : r_\Sigma(\ker \mathcal{H}^{s,p}) \to \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})$. Thus, $P_{(B,\Psi)} : \mathcal{T}^{s-1/p,p}_\Sigma \to \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})$ and $r_\Sigma P_{(B,\Psi)} = P^+_{(B,\Psi)}$. Moreover, from Corollary 3.18(ii), it follows that $P_{(B,\Psi)} : \mathcal{T}^{s-1/p,p}_\Sigma \to \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})$ and $r_\Sigma : \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}}) \to \mathcal{T}^{s-1/p,p}_\Sigma$ are inverse to each other.

(iv) We establish the smooth case, with the continuous case being exactly the same. It is easy to check that all the subspaces and operators involved in the construction of the maps in (ii) and (iii) vary smoothly with $(B(t),\Psi(t))$. Indeed, since $\mathcal{K}^{s-1,p}(Y)$ is a bundle, by Proposition 3.5, we can locally identify its fibers, i.e., the maps

$$\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} : \mathcal{K}^{s-1,p}_{(B(t),\Psi(t))} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}$$

are all isomorphisms for all $(B(t),\Psi(t))$ sufficiently $B^{s,p}(Y)$ close to a fixed (B_0,Ψ_0). Then restricting to t on a small interval for which this is the case, then we have $\ker \mathcal{H}_{(B(t),\Psi(t))} = \ker\big(\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} \mathcal{H}_{(B(t),\Psi(t))}\big)$, and $\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} \mathcal{H}^{s,p}_{(B(t),\Psi(t))} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}$ are all surjective for all t. From this, it follows that $\ker \mathcal{H}^{s,p}_{(B(t),\Psi(t))}$ varies smoothly, and since $\mathcal{J}^{s,p}_{(B(t),\Psi(t))} \in \ker \mathcal{H}^{s,p}_{(B(t),\Psi(t))}$ for all t, this implies $\ker(\mathcal{H}_{(B(t),\Psi(t))}|_{\mathcal{C}^{s,p}}$ vary smoothly. Indeed, one argues as in Lemma 3.9 for the continuity of $\ker \widetilde{\mathcal{H}}^{s,p}_{(B,\Psi)}$ with respect to (B,Ψ), only now we have in addition that all objects vary smoothly. Since $r_\Sigma : \ker(\mathcal{H}_{(B(t),\Psi(t))}|_{\mathcal{C}^{s,p}}) \to \mathcal{T}^{s-1/p,p}_\Sigma$ is an isomorphism onto its image for all t, it follows that $r_\Sigma(\ker \mathcal{H}^{s,p}_{(B(t),\Psi(t))})$ varies smoothly. Since this holds for all t on small intervals, it holds for all t along the whole path.

To prove the final statement, we observe that all the above methods apply to $\widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ and $\mathcal{H}^{t,q}_{(B,\Psi)}$ without modification in light of Lemma 3.9. See also Remark 3.10. □

We conclude this section with some important results that will be used later.

Lemma 3.19 *Let $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$, assume all the hypotheses of Theorem 3.13, and suppose (t,q) satisfies (3.92) or more generally (3.91). Then the space*

$$\mathsf{L}^{t-1/q,q}_{(B,\Psi)} := J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)}) \oplus B^{t-1/q,q}\Omega^0(\Sigma) \oplus 0 \tag{3.112}$$

is a complementary Lagrangian for $r(\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)})$ in $\widetilde{\mathcal{T}}^{t-1/q,q}_\Sigma$. The space $\mathsf{L}^{t-1/q,q}_{(B,\Psi)}$ varies continuously with $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$ (as long as $\mathcal{H}^{s,p}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is always surjective).

Proof By Theorem 3.13(i), $J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)})$ and $r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)})$ are complementary Lagrangians in $\mathcal{T}^{t-1/q,q}_\Sigma$. By Lemma 3.17(ii) and Corollary 3.18(i), it is now easy to see that (3.112) is a complement of $r(\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)})$ in $\widetilde{\mathcal{T}}^{t-1/q,q}_\Sigma$. Since $r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)})$ depends continuously on $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$ by Theorem 3.13(iv), the last statement follows. □

For $t > 1/q$, define

$$\widetilde{X}^{t,q}_{(B,\Psi)} = \{(b,\psi,\alpha) \in \widetilde{\mathcal{T}}^{t,q} : r(b,\psi,\alpha) \in J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)}) \oplus B^{t-1/q,q}\Omega^0(\Sigma) \oplus 0\}, \tag{3.113}$$

the subspace of $\widetilde{\mathcal{T}}^{t,q}$ whose boundary values lie in (3.112). Likewise, define

$$X^{t,q}_{(B,\Psi)} = \mathcal{C}^{t,q} \cap \widetilde{X}^{t,q}_{(B,\Psi)} \subset \mathcal{T}^{t,q}. \tag{3.114}$$

By the above lemma, the domains $\widetilde{X}^{t,q}_{(B,\Psi)}$ and $X^{t,q}_{(B,\Psi)}$ are such that their boundary values under r and r_Σ are complementary to the boundary values of $\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ and $\ker \mathcal{H}^{t,q}_{(B,\Psi)}$, respectively. Thus, we expect these domains to be ones on which the operators $\widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}$ and

$\mathcal{H}^{t,q}_{(B,\Psi)}$ are invertible elliptic operators. This is exactly what the following proposition tells us.

Proposition 3.20 *Let $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$ and assume all the hypotheses of Theorem 3.13. Let $t > 1/q$ and $q \geq 2$ satisfy (3.92) or more generally (3.91). Then the maps*

$$\widetilde{\mathcal{H}}_{(B,\Psi)} : \widetilde{X}^{t,q}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{t-1,q}, \tag{3.115}$$

$$\mathcal{H}_{(B,\Psi)} : X^{t,q}_{(B,\Psi)} \to \mathcal{K}^{t-1,q}_{(B,\Psi)} \tag{3.116}$$

are isomorphisms. Moreover, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{X}^{t,q}_{(B,\Psi)} & \xrightarrow{\widetilde{\mathcal{H}}_{(B,\Psi)}} & \widetilde{\mathcal{T}}^{t-1,q} \\ \uparrow & & \uparrow \\ X^{t,q}_{(B,\Psi)} & \xrightarrow{\mathcal{H}_{(B,\Psi)}} & \mathcal{K}^{t-1,q}_{(B,\Psi)} \end{array} \tag{3.117}$$

In particular, we can take $(t,q) = (s+1,p)$ in the above.

The previous statements all remain true if $\widetilde{X}^{t,q}_{(B,\Psi)}$ and $X^{t,q}_{(B,\Psi)}$ are replaced with $\widetilde{X}^{t,q}_{(B',\Psi')}$ and $X^{t,q}_{(B',\Psi')}$, respectively, for $(B',\Psi') \in \mathfrak{M}^{s,p}$ in a sufficiently small $B^{s,p}(Y)$ neighborhood of (B,Ψ).

Proof The map $\widetilde{\mathcal{H}}_{(B,\Psi)} : \widetilde{\mathcal{T}}^{t,q} \to \widetilde{\mathcal{T}}^{t-1,q}$ is surjective, by unique continuation, and by restricting to $\widetilde{X}^{t,q}_{(B,\Psi)}$, we have eliminated the kernel. Indeed, $r : \ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{t-1/q,q}_{\Sigma}$ is an isomorphism onto its image and $r(\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)}) \cap \mathsf{L}^{t,q}_{(B,\Psi)} = 0$, whence $\ker \widetilde{\mathcal{H}}^{t,q}_{(B,\Psi)} \cap \widetilde{X}^{t,q} = 0$. This proves (3.115) is an isomorphism. For (3.116), the same argument shows that (3.116) is injective. Indeed, $r_\Sigma : \ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{t,q}}) \to \mathcal{T}^{t-1/q,q}_{\Sigma}$ is injective by Corollary 3.18(ii) and Remark 3.10, and

$$r_\Sigma(\ker \mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{t,q}}) \cap r_\Sigma X^{t,q} = r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)}) \cap J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)})) = 0$$

by Theorem 3.13(i). It remains to show that (3.116) is surjective. We already know that $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{t,q} \to \mathcal{K}^{t-1,q}_{(B,\Psi)}$ is surjective by assumption. So given any $(a,\phi) \in \mathcal{T}^{s,p}$, we need to find a $(b,\psi) \in X^{t,q}_{(B,\Psi)}$ such that $\mathcal{H}_{(B,\Psi)}(b,\psi) = \mathcal{H}_{(B,\Psi)}(a,\phi)$. Without loss of generality, we can suppose $(a,\phi) \in \mathcal{C}^{t,q}$ by (3.38) and since $J^{t,q}_{(B,\Psi),t} \subseteq \ker \mathcal{H}^{t,q}_{(B,\Psi)}$. Since the condition $(b,\psi) \in X^{t,q}_{(B,\Psi)}$ imposes no restriction on the normal component of b at the boundary, we only need to make sure that $r_\Sigma(b,\psi) \in J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)})$. Since we have a decomposition

$$\mathcal{T}^{t-1/q,q}_{\Sigma} = r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)}) \oplus J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)}),$$

we can write $r_\Sigma(a,\phi) = (a_0,\phi_0) + (a_1,\phi_1)$ with respect to the above decomposition. Now let $(b,\psi) = (a,\phi) - P_{(B,\Psi)}(a_0,\phi_0)$, where $P_{(B,\Psi)}$ is the Poisson operator of $\mathcal{H}^{t,q}_{(B,\Psi)}$ with range equal $\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{t,q}})$ as given by Theorem 3.13. It follows that $(b,\psi) \in X^{t,q}_{(B,\Psi)}$, since $r_\Sigma(b,\psi) = (a_1,\phi_1) \in J_\Sigma r_\Sigma(\ker \mathcal{H}^{t,q}_{(B,\Psi)})$ and that $(b,\psi) \in \mathcal{C}^{t,q}$ since both (a,ϕ) and $P_{(B,\Psi)}(a_0,\phi_0)$ belong to $\mathcal{C}^{t,q}$. Thus, $(b,\psi) \in X^{t,q}_{(B,\Psi)}$ and we have $\mathcal{H}^{t,q}_{(B,\Psi)}(b,\psi) = \mathcal{H}^{t,q}_{(B,\Psi)}(a,\phi)$. So (3.116) is surjective, hence an isomorphism.

The commutativity of the diagram (3.117) now readily follows since (3.115) is an isomorphism which extends the isomorphism (3.116). Finally, for the last statement, we know that the space $\widetilde{X}^{t,q}_{(B,\Psi)}$ varies continuously with (B,Ψ) since the space $\mathsf{L}^{t-1/q,q}_{(B,\Psi)}$ varies continuously. Since $\mathcal{J}^{t,q}_{(B,\Psi),t} \subseteq \widetilde{X}^{t,q}_{(B,\Psi)}$ for all (B,Ψ), it follows that

$$X^{t,q}_{(B,\Psi)} = \Pi_{\mathcal{C}^{t,q}_{(B,\Psi)}}\{x \in \mathcal{T}^{t,q} : r(x) \in \mathsf{L}^{t-1/q,q}_{(B,\Psi)}\},$$

where $\Pi_{\mathcal{C}^{t,q}_{(B,\Psi)}}$ is the projection of $\mathcal{T}^{t,q}$ onto $\mathcal{C}^{t,q}_{(B,\Psi)}$ given by (3.38). From this, we see that $X^{t,q}_{(B,\Psi)}$ varies continuously since $\mathsf{L}^{t,q}_{(B,\Psi)}$ and $\mathcal{J}^{t,q}_{(B,\Psi),t}$ vary continuously. The continuity of $\tilde{X}^{t,q}_{(B,\Psi)}$ and $X^{t,q}_{(B,\Psi)}$ with respect to (B,Ψ) implies the last statement. □

The above proposition will be important when study the analytic properties of the spaces $\mathfrak{M}^{s,p}(Y)$ and $\mathcal{M}^{s,p}(Y)$ in the next section, where we will need to consider the inverse of the operator (3.116). The point is that by restricting the domain of the Hessian operator $\mathcal{H}_{(B,\Psi)}$, it becomes invertible and its inverse smooths by one derivative in a certain range of topologies depending on the regularity of the configuration (B,Ψ). Thus, the inverse of $\mathcal{H}_{(B,\Psi)}$ behaves like a pseudodifferential operator of order -1 in this range, which is what one would formally expect since $\mathcal{H}_{(B,\Psi)}$ is a first order operator. In particular, for (B,Ψ) smooth, we have the following corollary:

Corollary 3.21 *If $(B,\Psi) \in \mathfrak{M}$ is smooth, then for all $q \geq 2$ and $t > 1/q$, the maps*

$$\widetilde{\mathcal{H}}_{(B,\Psi)} : \widetilde{X}^{t,q}_{(B,\Psi)} \to \widetilde{\mathcal{T}}^{t-1,q}, \tag{3.118}$$

$$\mathcal{H}_{(B,\Psi)} : X^{t,q}_{(B,\Psi)} \to \mathcal{K}^{t-1,q}_{(B,\Psi)} \tag{3.119}$$

are isomorphisms.

4 The Space of Monopoles

Having studied the linear theory of the Hessian operators $\widetilde{\mathcal{H}}_{(B,\Psi)}$ and $\mathcal{H}_{(B,\Psi)}$ in the previous section, we now study the space of Besov monopoles $\mathfrak{M}^{s,p}(Y,\mathfrak{s})$ and $\mathcal{M}^{s,p}(Y,\mathfrak{s})$ on Y. Under suitable hypotheses, we show that these spaces are Banach manifolds and their local coordinate charts satisfy smoothing properties important for the analysis of Part IV. Moreover, we show that smooth monopoles are dense in the spaces $\mathfrak{M}^{s,p}(Y,\mathfrak{s})$ and $\mathcal{M}^{s,p}(Y,\mathfrak{s})$, so that these Banach manifolds are Besov completions of the smooth monopole spaces $\mathfrak{M}(Y,\mathfrak{s})$ and $\mathcal{M}(Y,\mathfrak{s})$, respectively. These analytic properties are crucial for the analysis in Part III.

Notation. Recall that $\mathcal{T}^{s,p}_{(B,\Psi)} = T_{(B,\Psi)}\mathfrak{C}^{s,p}(Y)$ is the tangent space to a configuration $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. Since all these tangent spaces are identical, in the previous section we worked within one fixed copy and called it $\mathcal{T}^{s,p}$. Now that we will work on the configuration space level, it is appropriate to keep track of the basepoint at times and we reintroduce this into our notation, though there really is no gain or loss of information by adding or dropping the basepoint from our notation.

Recall that we have fixed a spin^c structure $\mathfrak{s}$ from the start, which up to now, has not played any role in the analysis we have done. We now consider the following assumption:

$$c_1(\mathfrak{s}) \text{ is non-torsion or } H^1(Y,\Sigma) = 0. \tag{4.1}$$

The following lemma is the fundamental reason we make the above assumption:

Lemma 4.1 *Suppose (4.1) holds. Let $s > \max(3/p, 1/2)$. Then for every $(B,\Psi) \in \mathfrak{M}^{s,p}(Y,\mathfrak{s})$, we have $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is surjective.*

Proof There are two cases $s > 1$ and $s \leq 1$. We deal with the latter case, with the more regular case $s > 1$ being similar. So for $s < 1$, there are two main steps. First, we proceed as in the proof of Theorem 17.2 to show that any element in the cokernel of $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ must be more regular, in fact, it must lie in $\mathcal{K}^{s+1,p}_{(B,\Psi)}$. This follows because an element in the cokernel of $\mathcal{H}_{(B,\Psi)}$ satisfies an overdetermined elliptic boundary value problem, and thus we can bootstrap its regularity. Once we have enough regularity, we can integrate by parts, which shows that any element $(b,\psi) \in \mathcal{K}^{s+1,p}_{(B,\Psi)}$ in the cokernel of $\mathcal{H}^{s,p}_{(B,\Psi)}$ must satisfy $\mathcal{H}_{(B,\Psi)}(b,\psi) = 0$ and $r_\Sigma(b,\psi) = 0$. From here, the second step is to apply the unique continuation theorem, Corollary 17.5, to deduce that the cokernel of $\mathcal{H}^{s,p}_{(B,\Psi)}$ is zero.

For the first step, by Lemma 3.16, we know that $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ has closed range and finite dimensional cokernel. Let $(b,\psi) \in \mathcal{T}^{1-s,p'}_{(B,\Psi)}$, $p' = p/(p-1)$, be an element in the dual space of $\mathcal{K}^{s-1,p}_{(B,\Psi)}$ which annihilates $\operatorname{im} \mathcal{H}^{s,p}_{(B,\Psi)}$. Indeed, we have that $\mathcal{T}^{1-s,p'}_{(B,\Psi)}$ is the dual space of $\mathcal{T}^{s-1,p}_{(B,\Psi)}$ by Theorem 13.15. Next, we have the topological decomposition

$$\mathcal{T}^{1-s,p'}_{(B,\Psi)} = \mathcal{J}^{1-s,p'}_{(B,\Psi),t} \oplus \mathcal{K}^{1-s,p'}_{(B,\Psi)}. \tag{4.2}$$

This follows from the decomposition (3.37), since one can check that the map (3.43), by duality, is bounded on $\mathcal{T}^{1-s,p'}_{(B,\Psi)}$. More precisely, by our choice of s, we have the multiplication maps

$$\begin{aligned} B^{s,p}(Y) \times B^{s,p}(Y) &\to B^{s,p}(Y) \\ B^{s,p}(Y) \times B^{s-1,p}(Y) &\to B^{s-1,p}(Y), \end{aligned}$$

which by duality means that the multiplications

$$B^{s,p}(Y) \times B^{-s,p'}(Y) \to B^{-s,p'}(Y), \tag{4.3}$$

$$B^{s,p}(Y) \times B^{1-s,p'}(Y) \to B^{1-s,p'}(Y). \tag{4.4}$$

are also bounded. Thus, repeating the proof of (3.37) shows that there exists a bounded projection of $\mathcal{T}^{1-s,p'}_{(B,\Psi)}$ onto $\mathcal{J}^{1-s,p'}_{(B,\Psi)}$ through $\mathcal{K}^{1-s,p'}_{(B,\Psi)}$, for $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$. This proves (4.2).

Since $\mathcal{J}^{1-s,p'}_{(B,\Psi),t}$ and $\mathcal{K}^{s-1,p}_{(B,\Psi),t}$ annihilate each other, we see can choose our annihilating element $(b,\psi) \in \mathcal{K}^{1-s,p'}_{(B,\Psi)}$ since $\mathrm{im}\,(\mathcal{H}^{s,p}_{(B,\Psi)}) \subseteq \mathcal{K}^{s-1,p}_{(B,\Psi)}$. Moreover, the fact that (b,ψ) annihilates $\mathrm{im}\,(\mathcal{H}^{s,p}_{(B,\Psi)})$ also means that $\mathcal{H}_{(B,\Psi)}(b,\psi) = 0$ (weakly, i.e., as a distribution). Altogether then, we see that we have the weak equation

$$\widetilde{\mathcal{H}}_0(b,\psi) = (B - B_{\mathrm{ref}}, \Psi)\#(b,\psi). \tag{4.5}$$

Everything now proceeds as in the bootstrapping argument in Theorem 17.2, but with modifications since the multiplication term is not smooth. Because of the multiplication (4.4), we have $(B - B_{\mathrm{ref}}, \Psi)\#(b,\psi) \in \widetilde{\mathcal{T}}^{1-s,p'}$. By Theorem 15.19(i), $r_\Sigma(b,\psi) \in \mathcal{T}^{1-s-1/p',p'}_{\Sigma}$ is well-defined. Applying Green's formula to the symmetric operator $\mathcal{H}_{(B,\Psi)}$, we obtain for all $(a,\phi) \in \mathcal{T}$ that

$$\begin{aligned} 0 &= (\mathcal{H}_{(B,\Psi)}(a,\phi),(b,\psi))_{L^2(Y)} - ((a,\phi),\mathcal{H}_{(B,\Psi)}(b,\psi))_{L^2(Y)} \\ &= -\omega(r_\Sigma(a,\phi), r_\Sigma(b,\psi)). \end{aligned} \tag{4.6}$$

In the first line, we used that (b,ψ) annihilates $\mathrm{im}\,(\mathcal{H}_{(B,\Psi)})$ and $\mathcal{H}_{(B,\Psi)}(b,\psi) = 0$ (weakly). In the second line, we use that $r_\Sigma(b,\psi) \in \mathcal{T}^{1-s-1/p',p'}_{\Sigma}$ is well-defined. Since (4.6) holds for all $(a,\phi) \in \mathcal{T}$, we have $r_\Sigma(b,\psi) = 0$. This boundary condition together with (4.5) implies that we have an overdetermined elliptic boundary value problem (cf. Proposition 3.20, we have $r(b,\psi) \in 0 \oplus B^{1-s-1/p',p'}\Omega^0(\Sigma; i\mathbb{R}) \oplus 0$). By Theorem 15.19, this means we gain a derivative and so $(b,\psi) \in \mathcal{T}^{2-s,p'}_{(B,\Psi)}$. This implies $(B - B_0, \Psi)\#(b,\psi)$ is more regular than an element of $\mathcal{T}^{1-s-1/p',p'}$, and we can elliptic bootstrap again. We keep on boostrapping until we obtain $(b,\psi) \in \mathcal{T}^{s+1,p}_{(B,\Psi)}$, which is one derivative more regular than the maximum regularity of (4.5) since $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$. Thus, $(b,\psi) \in \mathcal{K}^{s+1,p}_{(B,\Psi)}$ is now a strong solution to $\mathcal{H}_{(B,\Psi)}(b,\psi) = 0$.

We can now use Corollary 17.5, since $\mathcal{K}^{s+1,p}_{(B,\Psi)} \subset \mathcal{K}^{1,2}_{(B,\Psi)}$, as $p \geq 2$. This theorem implies the following. Either $(b,\psi) = 0$, in which case the cokernel of $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is zero, or else $(B,\Psi) = (B,0)$ and $\psi \equiv 0$, $b \in H^1(Y,\Sigma; i\mathbb{R})$. In the former case, our map $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is surjective and we are done. For the latter case, we apply assumption (4.1). In case $c_1(\mathfrak{s})$ is non-torsion, $\det(\mathfrak{s})$ admits no flat connections, hence, we cannot have a reducible configuration $(B,\Psi) = (B,0)$ be a monopole, else B^t would be a flat connection on $\det(\mathfrak{s})$. In case $H^1(Y,\Sigma) = 0$, then we see $(b,\psi) = 0$ and the Hessian is surjective. This proves the lemma. □

Assumption: For the rest of Part I, we assume (4.1) holds.

So let us fix Y and $\mathfrak{s}$ satisfying (4.1), and write $\mathfrak{M}^{s,p} = \mathfrak{M}^{s,p}(Y,\mathfrak{s})$ and $\mathcal{M}^{s,p} = \mathfrak{M}^{s,p}(Y,\mathfrak{s})$ for short. The conclusion of the lemma guarantees that we have transversality for the monopole equations. This implies the following theorem:

Theorem 4.2 *For $s > \max(3/p, 1/2)$, $\mathfrak{M}^{s,p}$ and $\mathcal{M}^{s,p}$ are closed submanifolds of $\mathfrak{C}^{s,p}(Y)$.*

Proof For any smooth $(B,\Psi) \in \mathfrak{C}(Y)$, one can verify directly that $SW_3(B,\Psi) \in$

$\mathcal{K}_{(B,\Psi)}$.[19] Thus when $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, we have $SW_3(B,\Psi) \in \mathcal{K}^{s-1,p}_{(B,\Psi)}$, since the map $\mathbf{d}^*_{(B,\Psi)} : \mathcal{T}^{s-1,p}_{(B,\Psi)} \to B^{s-2,p}\Omega^0(Y;i\mathbb{R})$ is still bounded by our choice of s. Proceeding as in [21, Chapter 12], we can therefore think of $SW_3 : \mathfrak{C}^{s,p}(Y) \to \mathcal{K}^{s-1,p}(Y)$ as a section of the Banach bundle $\mathcal{K}^{s-1,p}(Y) \to \mathfrak{C}^{s,p}(Y)$ (see Proposition 3.5). The previous lemma shows that SW_3 is transverse to the zero section. More precisely, from Proposition 3.5, we have that $\mathcal{K}^{s-1,p}(Y) \to \mathfrak{C}^{s,p}(Y)$ is Banach bundle complementary to the bundle $\mathcal{J}^{s-1,p}_t(Y) \to \mathfrak{C}^{s,p}(Y)$, which means that for any configuration $(B_0,\Psi_0) \in \mathfrak{C}^{s,p}(Y)$, there exists a neighborhood $\mathfrak{U}$ of (B_0,Ψ_0) in $\mathfrak{C}^{s,p}(Y)$ such that

$$\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} : \mathcal{K}^{s-1,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)} \tag{4.7}$$

is an isomorphism for all $(B,\Psi) \in \mathfrak{U}$. Here, $\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} : \mathcal{T}^{s-1,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}$ is the projection through $\mathcal{J}^{s-1,p}_{(B_0,\Psi_0),t}$ given by (3.45). Thus, if $SW_3(B_0,\Psi_0) = 0$, we consider the map

$$f = \Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} SW_3 : \mathfrak{U} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)} \tag{4.8}$$

Then $f(B,\Psi) = 0$ if and only if $SW_3(B,\Psi) = 0$, and at such a monopole, we have

$$D_{(B,\Psi)}f = \Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} \mathcal{H}^{s,p}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}. \tag{4.9}$$

By Lemma 4.1, $\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B,\Psi)}$ is surjective, and so since (4.7) is an isomorphism, this means $D_{(B,\Psi)}f$ is surjective for all $(B,\Psi) \in \mathfrak{U}$. Thus, we can apply the implicit function theorem to conclude that $f^{-1}(0)$ is a submanifold of $\mathfrak{C}^{s,p}(Y)$. Since we can apply the preceding local model near every monopole, it follows that $\mathfrak{M}^{s,p} = SW_3^{-1}(0) \subset \mathfrak{C}^{s,p}(Y)$ is globally a smooth Banach submanifold. Lemma 3.2 implies that we have the product decomposition

$$\mathfrak{M}^{s,p} = \mathcal{G}^{s+1,p}_{\mathrm{id},\partial}(Y) \times \mathcal{M}^{s,p}. \tag{4.10}$$

Thus $\mathcal{M}^{s,p}$ is also a submanifold of $\mathfrak{C}^{s,p}(Y)$, since $\mathcal{G}^{s+1,p}_{\mathrm{id},\partial}(Y)$ is a smooth Banach Lie group by Lemma 3.1. The closedness of $\mathfrak{M}^{s,p}$ and $\mathcal{M}^{s,p}$ readily follows from the fact that these two spaces are defined as the zero set of equations. □

Remark 4.3 Note that we can take the open neighborhood $\mathfrak{U} \subset \mathfrak{C}^{s,p}(Y)$ of (B_0,Ψ_0) to contain a ball in the $L^2(Y)$ topology (so that $\mathfrak{U}$ is a very large open subset of $\mathfrak{C}^{s,p}(Y)$). Indeed, this is because $\mathcal{K}^{s-1,p}_{(B_1,\Psi_1)}$ and $\mathcal{J}^{s-1,p}_{(B_0,\Psi_0),t}$ are complementary for any (B_1,Ψ_1) in a sufficiently small $L^2(Y)$ neighborhood of (B_0,Ψ_0), and so the map (4.7) is an isomorphism

[19] This is no coincidence. On a closed-manifold Y, the Seiberg-Witten equations are the variational equations for the Chern-Simons-Dirac functional, see [21]. In other words, $SW_3(B,\Psi)$ is the gradient of the Chern-Simons-Dirac functional CSD, i.e., the differential of CSD at (B,Ψ) satisfies $D_{(B,\Psi)}CSD(b,\psi) = (SW_3(B,\Psi),(b,\psi))$ so that $SW_3(B,\Psi)$ vanishes precisely at the critical points of CSD. When ∂Y is nonempty, we still have $D_{(B,\Psi)}CSD(b,\psi) = (SW_3(B,\Psi),(b,\psi)) = 0$ for (b,ψ) vanishing on the boundary, in particular, for $(b,\psi) \in \mathcal{J}_{(B,\Psi),t}$. Since CSD is invariant under the gauge group $\mathcal{G}_{\mathrm{id},\partial}(Y)$, this means $(SW_3(B,\Psi),(b,\psi)) = 0$ for all $(b,\psi) \in \mathcal{J}_{(B,\Psi),t}$. So $SW_3(B,\Psi) \in \mathcal{K}_{(B,\Psi)}$, the orthogonal complement.

for $(B,\Psi)=(B_1,\Psi_1)$. To show this, it suffices to show that

$$\mathcal{K}^{s-1,p}_{(B_1,\Psi_1)}\cap\mathcal{J}^{s-1,p}_{(B_0,\Psi_0),t}=0. \tag{4.11}$$

Indeed, this will show that (4.7) injective. However, it must also be an isomorphism, since $\mathcal{K}^{s-1,p}_{(B,\Psi)}$ varies continuously with $(B,\Psi)\in\mathfrak{C}^{s,p}(Y)$ as a consequence of Proposition 3.5. Namely, since (4.7) is an isomorphism for $(B,\Psi)=(B_0,\Psi_0)$, then if it is injective for all $(B,\Psi)=(B(t),\Psi(t))$ along a path in $\mathfrak{C}^{s,p}(Y)$ joining (B_0,Ψ_0) to (B_1,Ψ_1), then it must also be an isomorphism for all such (B,Ψ).

We now show (4.11). Note that an element of $\mathcal{K}^{s-1,p}_{(B_1,\Psi_1)}\cap\mathcal{J}^{s-1,p}_{(B_0,\Psi_0),t}$ is determined by a $\xi\in B^{s,p}\Omega^0(Y;i\mathbb{R})$ that solves

$$\Delta\xi+\mathrm{Re}\,(\Psi_1,\Psi_0)\xi=0 \tag{4.12}$$
$$\xi|_\Sigma=0. \tag{4.13}$$

Using elliptic regularity for the Dirichlet Laplacian, we bootstrap the regularity of ξ to obtain $\xi\in B^{2,2}\Omega^2(Y;i\mathbb{R})$. Writing $\Delta+(\Psi_1,\Psi_0)=\Delta+|\Psi_0|^2+\mathrm{Re}\,(\Psi_1-\Psi_0,\Psi_0)$, we see that the operator $\Delta+(\Psi_1,\Psi_0)$ is a perturbation of the operator

$$\Delta+|\Psi_0|^2: B^{2,2}\Omega^0_t(Y;i\mathbb{R})\to L^2\Omega^0(Y;i\mathbb{R}),$$

whose domain $B^{2,2}\Omega^0_t(Y;i\mathbb{R})$ consists of those $\alpha\in B^{2,2}\Omega^0(Y;i\mathbb{R})$ such that $\alpha|_\Sigma=0$. We showed that this latter operator is invertible in the proof of Lemma 3.4. It follows that if the multiplication operator $\mathrm{Re}\,(\Psi_1-\Psi_0,\Psi_0)$ has small enough norm, as a map from $B^{2,2}(Y)$ to $L^2(Y)$, then the operator $\Delta+\mathrm{Re}\,(\Psi_1,\Psi_0)$ remains invertible and the only solution to (4.12)–(4.13) is $\xi=0$. We have

$$\begin{aligned}\|\mathrm{Re}\,(\Psi_1-\Psi_0,\Psi_0)\alpha\|_{L^2(Y)} &\le \|\Psi_1-\Psi_0\|_{L^2(Y)}\|\Psi_0\|_{L^\infty(Y)}\|\alpha\|_{L^\infty(Y)}\\ &\le C\|\Psi_1-\Psi_0\|_{L^2(Y)}\|\Psi_0\|_{B^{s,p}(Y)}\|\alpha\|_{B^{2,2}(Y)}.\end{aligned} \tag{4.14}$$

since both $B^{s,p}(Y)$ and $B^{2,2}(Y)$ embed into $L^\infty(Y)$. Hence, if $\|\Psi_1-\Psi_0\|_{L^2(Y)}$ is sufficiently small, we see that the only solution to (4.12)–(4.13) is $\xi=0$, which establishes (4.11).

Theorem 4.2 proves the first part of our main theorem. However, to better understand the analytic properties of these monopole spaces, we want to construct explicit charts for our manifolds $\mathfrak{M}^{s,p}$ and $\mathcal{M}^{s,p}$. Furthermore, we want to show that smooth monopoles are dense in these spaces. These properties are not only of interest in their own right but will be essential in Part III.

In a neighborhood of $(B,\Psi)\in\mathcal{M}^{s,p}$, the Banach manifolds $\mathfrak{M}^{s,p}$ and $\mathcal{M}^{s,p}$ are modeled on their tangent spaces at (B,Ψ), namely $\ker\mathcal{H}^{s,p}_{(B,\Psi)}$ and $\ker(\mathcal{H}_{(B,\Psi)}|_{\mathcal{C}^{s,p}})=\ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{T}^{s,p}})$, respectively. Moreover, the tangent space to our manifolds at (B,Ψ) are the range of operators which are "nearly pseudodifferential". Indeed, in the previous section, we constructed a Poisson operator $P_{(B,\Psi)}$ whose range is $\ker(\widetilde{\mathcal{H}}_{(B,\Psi)}|_{\mathcal{T}^{s,p}})$. Since this operator is constructed from the Calderon projection $P^+_{(B,\Psi)}$ and the Poisson operator $\tilde{P}^+_{(B,\Psi)}$ for the augmented Hessian $\widetilde{\mathcal{H}}_{(B,\Psi)}$, both of which differ from pseudodifferential operators by a compact operator, it is in this sense that $P_{(B,\Psi)}$ is close to being pseudodifferential.

Let $(B_1,\Psi_1),(B_0,\Psi_0)\in\mathfrak{C}^{s,p}(Y)$ and write $(b,\psi)=(B_1-B_0,\Psi_1-\Psi_0)$. Then we have the difference equation

$$SW_3(B_1,\Psi_1)-SW_3(B_0,\Psi_0)=\mathcal{H}_{(B_0,\Psi_0)}(b,\psi)+(\rho^{-1}(\psi\psi^*)_0,\rho(b)\psi), \tag{4.15}$$

which reflects the fact that SW_3 is a quadratic map. The linear part, is of course, given by the Hessian, and its quadratic part is just a pointwise multiplication map. Thus, we define the bilinear map

$$\begin{aligned} \mathsf{q}&:\mathcal{T}\times\mathcal{T}\to\mathcal{T}\\ \mathsf{q}((b_1,\psi_1),(b_2,\psi_2))&=\Big(\rho^{-1}(\psi_1\psi_2^*)_0,\frac{1}{2}\big(\rho(b_1)\psi_2+\rho(b_2)\psi_1\big)\Big) \end{aligned} \tag{4.16}$$

which as a quadratic function enters into the Seiberg-Witten map via (4.15). The map q extends to function space completions as governed by the multiplication theorems. Observe that q is a bounded map on $\mathcal{T}^{s,p}$ since $B^{s,p}(Y)$ is an algebra. This is key, because then the Seiberg-Witten map SW_3 is the sum of a first order differential operator and a zeroth order operator, and using Proposition 3.20, we have elliptic regularity for the linear part of the operator on suitable domains.

From these observations, we can prove the following important lemma which we will need to show that smooth monopoles are dense in $\mathfrak{M}^{s,p}$.

Lemma 4.4 *Let $s>\max(3/p,1/2)$. Let $(B_0,\Psi_0)\in\mathcal{M}^{s,p}$. Then $B^{s+1,p}(Y)$ configurations are dense in the affine space $(B_0,\Psi_0)+T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$.*

Proof Pick any smooth $(B_1,\Psi_1)\in\mathfrak{C}_C(Y)$ in Coulomb-gauge with respect to B_{ref}. Let $(b,\psi)=(B_1-B_0,\Psi_1-\Psi_0)$. Then from (4.15) together with the Coulomb-gauge condition, we have

$$\widetilde{\mathcal{H}}^{s,p}_{(B_0,\Psi_0)}(b,\psi)=SW_3(B_1,\Psi_1)-\mathsf{q}((b,\psi),(b,\psi)), \tag{4.17}$$

where on the right-hand side the first term is smooth and the second term is in $\mathcal{T}^{s,p}$. Applying Proposition 3.20 with $(t,q)=(s+1,p)$, we see that $(b,\psi)\in(b',\psi')+\ker(\widetilde{\mathcal{H}}_{(B_0,\Psi_0)}|_{\mathcal{T}^{s,p}})$ for some $(b',\psi')\in X^{s+1,p}_{(B_0,\Psi_0)}\subseteq\mathcal{T}^{s+1,p}$. In other words, if we invert $\widetilde{\mathcal{H}}^{s,p}_{(B_0,\Psi_0)}$ in (4.17), we find that (b,ψ) is equal to a smoother element (b',ψ'), modulo an element of the kernel of $\widetilde{\mathcal{H}}^{s,p}_{(B_0,\Psi_0)}|_{\mathcal{T}^{s,p}}$. It remains to show that $B^{s+1,p}(Y)$ configurations are dense in the latter space. First, we have $B^{s+1,p}(Y)$ configurations are dense in $\ker\widetilde{\mathcal{H}}^{s,p}_{(B_0,\Psi_0)}\subset\widetilde{\mathcal{T}}^{s,p}$ by Corollary 15.17 and Lemma 3.9. Similarly, $B^{s+1,p}(Y)$ configurations are dense in Γ_0, the subspace given by (3.105). This follows from the construction of Γ_0. First, we have Γ_0 is a graph of the map Θ_0, which is defined over $\ker\Delta\subseteq B^{s,p}\Omega^0(Y;i\mathbb{R})$, and smooth configurations are dense in $\ker\Delta$ by Corollary 15.17. We now apply Proposition 3.20 with $(t,q)=(s+1,p)$, since the map Θ_0 is defined by inverting the Hessian. Altogether, we see that $B^{s+1,p}(Y)$ configurations are dense in Γ_0. Because of the decomposition (3.105), it now follows from the density of $B^{s+1,p}(Y)$ configurations in $\ker\widetilde{\mathcal{H}}^{s,p}_{(B_0,\Psi_0)}$ and Γ_0 that $B^{s+1,p}(Y)$ configurations are dense in $\ker(\widetilde{\mathcal{H}}_{(B_0,\Psi_0)}|_{\mathcal{T}^{s,p}})=T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$.

Altogether, we have shown that $(B_0,\Psi_0)+T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}=(B_1,\Psi_1)+(b',\psi')+T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$, where (B_1,Ψ_1) is smooth, $(b',\psi')\in\mathcal{T}^{s+1,p}$, and $B^{s+1,p}(Y)$ configurations are dense in

$T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$. This proves the lemma. □

From Theorem 3.13, given $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$, we have a projection $\pi_{(B_0,\Psi_0)} = P_{(B_0,\Psi_0)} r_\Sigma : \mathcal{T}^{s,p}_{(B_0,\Psi_0)} \to T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ onto the tangent space $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ for any $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$. Thus, locally $\mathfrak{M}^{s,p}$ is the graph of a map from $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ to any complementary subspace in $\mathcal{T}^{s,p}_{(B_0,\Psi_0)}$. We wish to describe the analytic properties of this local graph model in more detail. First, we record the following simple lemma which describes for us natural complementary subspaces for $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$.

Lemma 4.5 *Let $s > \max(3/p, 1/2)$. Given any $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$, we have the direct sum decomposition*

$$\mathcal{T}^{s,p}_{(B_0,\Psi_0)} = T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p} \oplus X^{s,p}_{(B,\Psi)} \tag{4.18}$$

for any $(B, \Psi) \in \mathfrak{M}^{s,p}$ sufficiently $B^{s,p}(Y)$ close to (B_0, Ψ_0), where $X^{s,p}_{(B,\Psi)}$ is defined as in (3.114).

Proof By Lemma 4.1, $\mathcal{H}_{(B_0,\Psi_0)} : \mathcal{T}^{s,p}_{(B_0,\Psi_0)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}$ is surjective. Thus, (4.18) follows readily from $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p} = \ker \mathcal{H}^{s,p}_{(B_0,\Psi_0)}$ and $\mathcal{H}_{(B_0,\Psi_0)} : X^{s,p}_{(B,\Psi)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}$ being an isomorphism by Proposition 3.20. Note also that $X^{s,p}_{(B,\Psi)}$ is the kernel of the projection $\pi_{(B,\Psi)} : \mathcal{T}^{s,p} \to T_{(B,\Psi)}\mathcal{M}^{s,p}$. □

Using any one of above complementary subspaces for $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ (we will always use $X^{s,p}_{(B_0,\Psi_0)}$ for simplicity), we can describe the Banach manifold $\mathfrak{M}^{s,p}$ locally as follows. In the proof of Theorem 4.2, we introduced the local defining function f in (4.8) on a neighborhood $\mathfrak{U} \subset \mathfrak{C}^{s,p}(Y)$ so that $\mathfrak{M}^{s,p} \cap \mathfrak{U} = f^{-1}(0)$. In other words, we used the implicit function theorem for f to obtain $\mathfrak{M}^{s,p}$. On the other hand, we can describe $\mathfrak{M}^{s,p}$ in an equivalent way using the inverse function theorem, as in the framework of Theorem 20.2, whereby $\mathfrak{M}^{s,p}$ is given locally by the preimage of an open set under diffeormophism rather than the preimage of a regular value of a surjective map. This means we need to construct a *local straightening map* as in Definition 20.3. Following the same ansatz in Theorem 20.2, we have the following:

Lemma 4.6 *Let $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$, and let $X = \mathcal{T}^{s,p}_{(B_0,\Psi_0)}$, $X_0 = T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$, and $X_1 = X^{s,p}_{(B_0,\Psi_0)}$. We have $X = X_0 \oplus X_1$ and define the map*

$$\begin{aligned} F_{(B_0,\Psi_0)} &: X_0 \oplus X_1 \to X \\ x = (x_0, x_1) &\mapsto x_0 + (\mathcal{H}_{(B_0,\Psi_0)}|_{X_1})^{-1}\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} SW_3\big((B_0, \Psi_0) + x\big). \end{aligned} \tag{4.19}$$

(i) Then $F_{(B_0,\Psi_0)}(0) = 0$, $D_0F_{(B_0,\Psi_0)} = 0$, and $F_{(B_0,\Psi_0)}$ is a local diffeomorphism in a $B^{s,p}(Y)$ neighborhood of 0.

(ii) There exists an open set $V \subset X$ containing 0 such that for any $x \in V$, we have $(B_0, \Psi_0) + x \in \mathfrak{M}^{s,p}$ if and only if $F_{(B_0,\Psi_0)}(x) \in X_0$. We can choose V to contain an $L^2(Y)$ ball, i.e., there exists a $\delta > 0$ such that

$$V \supseteq \{x \in X : \|x\|_{L^2(Y)} < \delta\}.$$

Furthermore, we can choose $\delta = \delta(B_0, \Psi_0)$ uniformly for all (B_0, Ψ_0) in a sufficiently small $L^\infty(Y)$ neighborhood of any configuration in $\mathfrak{M}^{s,p}$.

(iii) If $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$ then for any $x \in V \cap \mathcal{C}^{s,p}_{(B_0,\Psi_0)}$, we have $(B_0, \Psi_0) + x \in \mathcal{M}^{s,p}$ if and only if $F_{(B_0,\Psi_0)}(x) \in X_0 \cap \mathcal{C}^{s,p}_{(B_0,\Psi_0)}$.

Proof (i) We have $F_{(B_0,\Psi_0)}(0) = 0$ since $SW_3(B_0, \Psi_0) = 0$. Furthermore, the differential of $F_{(B_0,\Psi_0)}$ at 0 is the identity map by construction; more explicitly,

$$\begin{aligned}(D_0 F_{(B_0,\Psi_0)})(x) &= x_0 + (\mathcal{H}_{(B_0,\Psi_0)}|_{X_1})^{-1}\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}}\mathcal{H}_{(B_0,\Psi_0)}(x)\\ &= x_0 + (\mathcal{H}_{(B_0,\Psi_0)}|_{X_1})^{-1}\mathcal{H}_{(B_0,\Psi_0)}(x) = x_0 + x_1.\end{aligned}$$

So by the inverse function theorem, $F_{(B_0,\Psi_0)}$ is a local diffeomorphism in a $B^{s,p}(Y)$ neighborhood of 0.

(ii) Observe that $F_{(B_0,\Psi_0)}(x) \in X_0$ if and only if the second term of (4.19), which lies in X_1, vanishes. Let $(B, \Psi) = (B_0, \Psi_0)+x$. Then for x in a small $L^2(Y)$ neighborhood of $0 \in X$, call it V, we know by Remark 4.3 that (4.7) is an isomorphism. Since $SW_3\big((B_0, \Psi_0) + x\big) \in \mathcal{K}^{s-1,p}_{(B,\Psi)}$ and $(\mathcal{H}_{(B_0,\Psi_0)}|_{X_1})^{-1}$ is an isomorphism, it follows that the second term of (4.19) vanishes if and only if $SW_3\big((B_0, \Psi_0) + x\big)$ vanishes, i.e., if and only if $(B_0, \Psi_0) + x \in \mathfrak{M}^{s,p}$. Equation (4.14) shows that the size of this $L^2(Y)$ ball depends only on $\|\Psi_0\|_{L^\infty(Y)}$, and this implies the continuity statement for δ.

(iii) Since $X_1 \subset \mathcal{C}^{s,p}_{(B_0,\Psi_0)}$ by (3.114), we have that $F_{(B_0,\Psi_0)}(x) \in \mathcal{C}^{s,p}_{(B_0,\Psi_0)}$ if and only if $x_0 \in \mathcal{C}^{s,p}_{(B_0,\Psi_0)}$. Then (iii) now follows from the previous steps via intersection with $\mathcal{C}^{s,p}_{(B_0,\Psi_0)}$. □

Thus, the map $F_{(B_0,\Psi_0)}$ in the above lemma is a local straightening map for $\mathfrak{M}^{s,p}$ (where we translate by the basepoint (B_0, Ψ_0) so that we can regard $\mathfrak{M}^{s,p}$ as living inside the Banach space $\mathcal{T}^{s,p}_{(B_0,\Psi_0)}$) such that its restriction to $\mathcal{C}^{s,p}_{(B_0,\Psi_0)}$ yields a local straightening map for $\mathcal{M}^{s,p}$ if $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$. In Theorem 4.8, we will show, in the precise sense of Definition 20.3 that $F_{(B_0,\Psi_0)}$ is a local straightening map for $\mathcal{M}^{s,p}$ within a "large" neighborhood of (B_0, Ψ_0), where large means that the open set contains a ball in a topology weaker than the ambient $B^{s,p}(Y)$ topology. First, we need another important lemma, which allows us to redefine $F_{(B_0,\Psi_0)}$ on weaker function spaces:

Lemma 4.7 *Let $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$ for $s > \max(3/p, 1/2)$.*

(i) If $x \in \mathcal{T}^{s,p}_{(B_0,\Psi_0)}$, then we can write $F_{(B_0,\Psi_0)}(x)$ as

$$F_{(B_0,\Psi_0)}(x) = x + (\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}})^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}\mathsf{q}(x, x), \tag{4.20}$$

$$=: x + Q_{(B_0,\Psi_0)}(x, x), \tag{4.21}$$

where q is the quadratic multiplication map given by (4.16).

(ii) The map $(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}})^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : \mathcal{T}^{s,p} \to \mathcal{T}^{s+1,p}$ extends to a bounded map

$$(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}})^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : L^q\mathcal{T} \to H^{1,q}\mathcal{T} \tag{4.22}$$

for any $1 < q < \infty$.

(iii) Let $3 \leq q \leq \infty$. *For* $x \in L^q\mathcal{T}_{(B_0,\Psi_0)}$, *define* $F_{(B_0,\Psi_0)}(x)$ *by (4.21). Then* $F_{(B_0,\Psi_0)} : L^q\mathcal{T}_{(B_0,\Psi_0)} \to L^q\mathcal{T}_{(B_0,\Psi_0)}$ *is a local diffeomorphism in a* $L^q(Y)$ *neighborhood of* 0.

Proof (i) With $x = (x_0, x_1)$ as in Lemma 4.6, we have

$$\begin{aligned} F(x) &= x_0 + (\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s,p}_{(B_0,\Psi_0)}})^{-1}\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} SW_3\big((B_0,\Psi_0)+x\big) \\ &= x_0 + (\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s,p}_{(B_0,\Psi_0)}})^{-1}\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} \big(\mathcal{H}_{(B_0,\Psi_0)}(x) + \mathsf{q}(x,x)\big) \\ &= x_0 + x_1 + (\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s,p}_{(B_0,\Psi_0)}})^{-1}\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}} \mathsf{q}(x,x). \end{aligned} \tag{4.23}$$

Next, since $B^{s,p}(Y)$ is an algebra, then $\mathsf{q}(x,x) \in \mathcal{T}^{s,p}$. It follows that in (4.23), we may replace $\Pi_{\mathcal{K}^{s-1,p}_{(B_0,\Psi_0)}}$ with $\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}$. From Proposition 3.20, we know that we have isomorphisms

$$\begin{aligned} \mathcal{H}_{(B_0,\Psi_0)} &: X^{s,p}_{(B_0,\Psi_0)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)} \\ \mathcal{H}_{(B_0,\Psi_0)} &: X^{s+1,p}_{(B_0,\Psi_0)} \to \mathcal{K}^{s,p}_{(B_0,\Psi_0)}, \end{aligned}$$

from which it follows that if $y \in \mathcal{K}^{s,p}_{(B_0,\Psi_0)}$, then $\mathcal{H}^{-1}_{(B_0,\Psi_0)}(y) \in X^{s+1,p}_{(B_0,\Psi_0)}$. The decomposition (4.20) now follows.

(ii) First, we note that Lemma 3.4 extends to Sobolev spaces, since its proof, which involves studying elliptic boundary value problems, carries over verbatim to Sobolev spaces (see Section 15) so long as the requisite function space multiplication works out. In this case, we want $\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}$ to yield a bounded map on $L^q\mathcal{T}$, in which case, the bounded multiplications that we want are the boundedness of

$$B^{s,p}(Y) \times H^{1,q}(Y) \to L^q(Y) \tag{4.24}$$

$$B^{s,p}(Y) \times L^q(Y) \to H^{-1,q}(Y), \tag{4.25}$$

cf. (3.28) and (3.29). However, these are straightforward, because we have the embedding $B^{s,p}(Y) \hookrightarrow L^\infty(Y)$, and we have the obvious bounded multiplication $L^\infty(Y) \times L^q(Y) \to L^q(Y)$, which therefore trivially imply the above multiplications.

From this, it remains to show that $(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}})^{-1}$ extends to a bounded map $L^q\mathcal{T} \to H^{1,q}\mathcal{T}$. However, the exact same considerations show that this is the case due to the boundedness of the above multiplication maps.

(iii) We have a bounded multiplication map $L^q(Y) \times L^q(Y) \to L^{q/2}(Y)$, and for $q \geq 3$, we have the Sobolev embedding $H^{1,q/2}(Y) \hookrightarrow L^q(Y)$. Hence (ii) implies that the map $F_{(B_0,\Psi_0)}$ is bounded on $L^q\mathcal{T}$ for $q \geq 3$. Since $F_{(B_0,\Psi_0)}(0) = 0$ and $D_0F_{(B_0,\Psi_0)} = \mathrm{id}$, the inverse function theorem implies $F_{(B_0,\Psi_0)}$ is a local diffeomorphism in a $L^q(Y)$ neighborhood of 0. □

Thus, from now on, we may work with the expression (4.21) for $F_{(B_0,\Psi_0)}$ since it coincides with (4.19) when the latter is well-defined.

Given the local straightening map $F_{(B_0,\Psi_0)}$ and the various properties it obeys above, we now import the abstract point of view in Section 20 into our particular situation to construct charts for $\mathfrak{M}^{s,p}$. This gives us the following picture for a neighborhood of the monopole space $\mathfrak{M}^{s,p}$. At any $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$, letting $X_1 = X^{s,p}_{(B_0,\Psi_0)}$ be a complement of $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ as in Lemma 4.5, then near (B_0, Ψ_0), the space $\mathfrak{M}^{s,p}(Y)$ is locally the graph of a map, which we denote by $E^1_{(B_0,\Psi_0)}$, from a neighborhood of 0 in $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ to X_1. The local chart map we obtain in this way for $\mathfrak{M}^{s,p}(Y)$ is precisely the induced chart map of the local straightening map $F_{(B_0,\Psi_0)}$ above, in the sense of Definition 20.4. In addition, we show that the map $E^1_{(B_0,\Psi_0)}$ is smoothing, due to the fact that the lower order term $Q_{(B_0,\Psi_0)}$ occurring in $F_{(B_0,\Psi_0)}$, as defined in (4.21), is smoothing, i.e., it maps $\mathcal{T}^{s,p}_{(B_0,\Psi_0)}$ to $\mathcal{T}^{s+1,p}_{(B_0,\Psi_0)}$. Moreover, for any $q > 3$, we show that $F_{(B_0,\Psi_0)}$ is a local straightening map in some $L^q(Y)$ neighborhood of (B_0, Ψ_0). Consequently, the induced chart maps we obtain yield charts for $L^q(Y)$ neighborhoods of $\mathfrak{M}^{s,p}$, which are large neighborhoods when viewed within the ambient $B^{s,p}(Y)$ topology. This latter property will be very important in Part III, and it is the analog of how the local Coulomb slice theorems for nonabelian gauge theory allow for gauge fixing within large neighborhoods (i.e., neighborhoods defined with respect to a weak norm) of a reference connection (see e.g. [53, Theorem 8.1]).[20]

We have the following theorem:

Theorem 4.8 *Assume $s > \max(3/p, 1/2)$.*

(i) Let $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$ and $X_1 = X^{s,p}_{(B_0,\Psi_0)}$ be a complement of $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ in $\mathcal{T}^{s,p}_{(B_0,\Psi_0)}$. Then there exists a neighborhood U of $0 \in T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$ and a map $E^1_{(B_0,\Psi_0)} : U \to X_1$ such that the map

$$\begin{aligned} E_{(B_0,\Psi_0)} : U &\to \mathfrak{M}^{s,p} \\ x &\mapsto (B_0, \Psi_0) + x + E^1_{(B_0,\Psi_0)}(x) \end{aligned} \tag{4.26}$$

is a diffeomorphism of U onto an open neighborhood of (B_0, Ψ_0) in $\mathfrak{M}^{s,p}$. We have $E^1_{(B_0,\Psi_0)}(0) = 0$, $D_0E^1_{(B_0,\Psi_0)} = 0$, and furthermore, the map $E^1_{(B_0,\Psi_0)}$ smooths by one derivative, i.e., $E^1_{(B_0,\Psi_0)}(x) \in \mathcal{T}^{s+1,p}_{(B_0,\Psi_0)}$ for all $x \in U$.

(ii) Let $q > 3$. We can choose U such that both U and its image $E_{(B_0,\Psi_0)}(U)$ contain $L^q(Y)$ neighborhoods, i.e., there exists a $\delta > 0$, depending on (B_0, Ψ_0), such that

$$\begin{aligned} U &\supseteq \{x \in T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p} : \|x\|_{L^q(Y)} < \delta\} \\ E_{(B_0,\Psi_0)}(U) &\supseteq \{(B, \Psi) \in \mathcal{M}^{s,p} : \|(B, \Psi) - (B_0, \Psi_0)\|_{L^q(Y)} < \delta\}. \end{aligned}$$

The constant δ can be chosen uniformly in (B_0, Ψ_0), for all (B_0, Ψ_0) in a sufficiently small $L^\infty(Y)$ neighborhood of any configuration in $\mathfrak{M}^{s,p}$.

(iii) If $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$, then the map $E_{(B_0,\Psi_0)}$ restricted to $U \cap \mathcal{C}^{s,p}_{(B_0,\Psi_0)}$ is a diffeomorphism onto a neighborhood of (B_0, Ψ_0) in $\mathcal{M}^{s,p}$.

(iv) The smooth monopole spaces $\mathfrak{M}$ and $\mathcal{M}$ are dense in $\mathfrak{M}^{s,p}$ and $\mathcal{M}^{s,p}$, respectively.

[20] Such gauge fixing properties are important for issues related to compactness, since in proving a compactness theorem, one considers a sequence of configurations that are bounded in some norm, hence strongly convergent along a subsequence but with respect to a weaker norm. If one wants to gauge fix the elements in the convergent subsequence, one therefore needs a gauge fixing theorem on balls defined with respect to the weaker norm.

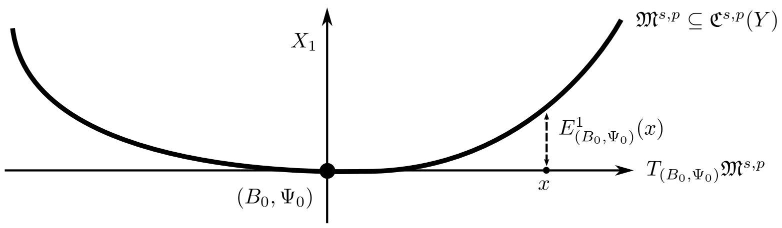

Figure I-1: A local chart map for $\mathfrak{M}^{s,p}$ at (B_0, Φ_0).

Proof (i-ii) Given $q > 3$, consider $F_{(B_0,\Psi_0)}$ as defined by (4.21). By Lemma 4.7(iii), $F_{(B_0,\Psi_0)}$ has a local inverse $F^{-1}_{(B_0,\Psi_0)}$ defined in an L^q neighborhood, call it V_q, of $0 \in L^q\mathcal{T}$. First, we show that $F^{-1}_{(B_0,\Psi_0)}$ (equivalently, $F_{(B_0,\Psi_0)}$) is regularity preserving, namely, that $F^{-1}_{(B_0,\Psi_0)}(x) \in \mathcal{T}^{s,p}_{(B_0,\Psi_0)}$ if and only if $x \in \mathcal{T}^{s,p}_{(B_0,\Psi_0)} \cap V_q$. In one direction, suppose $F^{-1}_{(B_0,\Psi_0)}(x)$ belongs to $\mathcal{T}^{s,p}_{(B_0,\Psi_0)}$. Then since $F_{(B_0,\Psi_0)}$ is continuous on $\mathcal{T}^{s,p}_{(B_0,\Psi_0)}$, then $x = F_{(B_0,\Psi_0)}(F^{-1}_{(B_0,\Psi_0)}(x)) \in \mathcal{T}^{s,p}_{(B_0,\Psi_0)}$. In the other direction, if $x \in \mathcal{T}^{s,p}_{(B_0,\Psi_0)}$, we apply (4.21) to obtain

$$F^{-1}_{(B_0,\Psi_0)}(x) = x - Q_{(B_0,\Psi_0)}(F^{-1}_{(B_0,\Psi_0)}(x), F^{-1}_{(B_0,\Psi_0)}(x)).$$

A priori, we only know that $F^{-1}_{(B_0,\Psi_0)}(x) \in L^q\mathcal{T}_{(B_0,\Psi_0)}$. However, in the above, we have $x \in \mathcal{T}^{s,p}_{(B_0,\Psi_0)}$ and $Q_{(B_0,\Psi_0)}(x) \in H^{1,q/2}\mathcal{T}$ by Lemma 4.7. When $q > 3$, then $Q_{(B_0,\Psi_0)}$ always gains for us regularity, and so we can bootstrap the regularity of $F^{-1}_{(B_0,\Psi_0)}(x)$ until it has the same regularity as x. Thus, this shows that $x \in \mathcal{T}^{s,p}$ if and only if $F^{-1}_{(B_0,\Psi_0)}(x) \in \mathcal{T}^{s,p}$.

Shrink V_q if necessary so that $V_q \cap \mathcal{T}^{s,p} \subset V_2$, where V_2 is defined to be the open set in Lemma 4.6(ii). This is possible since V_2 contains an $L^2(Y)$ ball and $q > 2$. Then if we let $V = V_q \cap \mathcal{T}^{s,p}$, then V satisfies the key property of Lemma 4.6(ii), namely if $x \in V$, then $(B_0, \Psi_0) + x \in \mathcal{M}^{s,p}$ if and only if $F(x) \in X_0$. The key step we have done here is that we have shown that $F^{-1}_{(B_0,\Psi_0)}$ is well-defined on the open set V, so that $F_{(B_0,\Psi_0)}$ becomes a local straightening map for $\mathfrak{M}^{s,p}$ within the neighborhood V of $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$. Indeed, with just Lemma 4.6, we would only know that $F_{(B_0,\Psi_0)}$ is a straightening map for a small $B^{s,p}(Y)$ neighborhood of $(B_0, \Psi_0) \in \mathfrak{M}^{s,p}$, which is what we get when we apply the inverse function theorem for $F_{(B_0,\Psi_0)}$ as a map on $\mathcal{T}^{s,p}$. Here, by rewriting $F_{(B_0,\Psi_0)}$ in Lemma 4.7 in a way that makes sense on L^q, we get an L^q open set on which we have the inverse $F^{-1}_{(B_0,\Psi_0)}$. The smoothing property of $Q_{(B_0,\Psi_0)}$ allows us to conclude the regularity preservation property of $F^{-1}_{(B_0,\Psi_0)}$, i.e., it preserves the $B^{s,p}(Y)$ topology, so that altogether, the map $F_{(B_0,\Psi_0)}$ is a straightening map for $\mathfrak{M}^{s,p}$ on the large open set $V \subset \mathcal{T}^{s,p}$.

Once we have the local straightening map $F_{(B_0,\Psi_0)}$, the construction of induced chart maps for $\mathfrak{M}^{s,p}$ now follows from the general picture described in the appendix. Letting $U = F(V) \cap X_0$, the map $E_{(B_0,\Psi_0)}$ is given by

$$E_{(B_0,\Psi_0)}(x) = (B_0, \Psi_0) + F^{-1}_{(B_0,\Psi_0)}(x), \qquad x \in U. \tag{4.27}$$

The map $E^1_{(B_0,\Psi_0)}(x)$ is just the nonlinear part of $E_{(B_0,\Psi_0)}(x)$, and it is given by

$$E^1_{(B_0,\Psi_0)}(x) = F^{-1}_{(B_0,\Psi_0)}(x) - x \tag{4.28}$$
$$= -Q_{(B_0,\Psi_0)}(F^{-1}_{(B_0,\Psi_0)}(x), F^{-1}_{(B_0,\Psi_0)}(x)), \qquad x \in U. \tag{4.29}$$

The smoothing property of $E^1_{(B_0,\Psi_0)}$ now readily follows from the smoothing property of $Q_{(B_0,\Psi_0)}$. By construction, U contains an $L^q(Y)$ ball since V does. This implies $E_{(B_0,\Psi_0)}(U)$ contains an $L^q(Y)$ neighborhood of $(B_0,\Psi_0) \in \mathfrak{M}^{s,p}(Y)$, since $\mathfrak{M} \cap ((B_0,\Psi_0)+V) = E_{(B_0,\Psi_0)}(U)$.

Finally, the local uniform dependence of δ can be seen as follows. First, the constant δ of Lemma 4.6 can be chosen uniformly for (B_0,Ψ_0) in a small $L^\infty(Y)$ neighborhood of any configuration in $\mathfrak{M}^{s,p}$. Next, the map $F_{(B_0,\Psi_0)} : L^q\mathcal{T} \to \mathcal{L}^q\mathcal{T}$ varies continuously as (B_0,Ψ_0) varies in the $L^\infty(Y)$ topology. It follows from the construction of V that we can find a fixed δ such that V contains a δ-ball in the $L^q(Y)$ topology as (B_0,Ψ_0) varies inside a small $L^\infty(Y)$ ball. We have now established all statements in (i-ii).

(iii) This follows from the above and Lemma 4.6(iii).

(iv) By Lemma 4.4 and the smoothing property of $E^1_{(B_0,\Psi_0)}$, we have that $\mathcal{M}^{s+1,p}$ is dense in $\mathcal{M}^{s,p}$. Iterating this in s, we see that $\mathcal{M}$ is dense in $\mathcal{M}^{s,p}$. Since smooth gauge transformations are dense in the space of gauge transformations, it follows from the decomposition $\mathfrak{M}^{s,p} = \mathcal{G}^{s+1,p}_{\mathrm{id},\partial}(Y) \times \mathcal{M}^{s,p}$ that $\mathfrak{M}$ is dense in $\mathfrak{M}^{s,p}$ as well. □

Retracing through the steps in the proof of Theorem 4.8, one sees that the chart maps for $\mathfrak{M}^{s,p}$ define bounded maps on weaker function spaces. This allows us to extend these chart maps to $L^q(Y)$ balls inside the closures of the tangent spaces to $\mathfrak{M}^{s,p}$ in weaker topologies. This yields for us the following important corollary:

Corollary 4.9 *Let $(B_0,\Psi_0) \in \mathfrak{M}^{s,p}$. Let $1/p \le t \le s$ and pick $q \ge 3$ according to the following: for $t = 1/p$, set $q = 3$; else for $t > 1/p$, choose $q > 3$. Consider the open subset*

$$U^{t,p} = \{x \in B^{t,p}(T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}) : \|x\|_{L^q(Y)} < \delta\}$$

of $B^{t,p}(T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p})$, the $B^{t,p}$ closure of $T_{(B_0,\Psi_0)}\mathfrak{M}^{s,p}$.

(i) For δ sufficiently small, $E_{(B_0,\Psi_0)}$ extends to a bounded map $E_{(B_0,\Psi_0)} : U^{t,p} \to \mathfrak{C}^{t,p}(Y)$. It is a diffeomorphism onto its image and is therefore a submanifold of $\mathfrak{C}^{t,p}(Y)$ contained in $\mathfrak{M}^{t,p}$.

(ii) The constant δ can be chosen uniformly for (B_0,Ψ_0) in a sufficiently small $L^\infty(Y)$ ball around any configuration of $\mathfrak{M}^{s,p}$.

The corresponding results hold also for $\mathcal{M}^{s,p}$. Finally, all the previous statements hold with the $B^{t,p}(Y)$ topology replaced with the $H^{t,p}(Y)$ topology.

Proof We only do the lowest regularity case $t = 1/p$, since the case $t > 1/p$ is simpler and handled in a similar way. For $t = 1/p$, then in trying to mimic the proof of Theorem 4.8, we show that the map $F_{(B_0,\Psi_0)} : L^3\mathcal{T}_{(B_0,\Psi_0)} \to L^3\mathcal{T}_{(B_0,\Psi_0)}$ preserves $B^{1/p,p}(Y)$ regularity on a small $L^3(Y)$ neighborhood of 0.

In one direction, starting with $x \in \mathcal{T}^{1/p,p}_{(B_0,\Psi_0)}$, we want to show that $F_{(B_0,\Psi_0)}(x) \in \mathcal{T}^{1/p,p}_{(B_0,\Psi_0)}$. This means we must show that $Q_{(B_0,\Psi_0)}$ is bounded on $\mathcal{T}^{t,p}$. We have the embedding $B^{1/p,p}(Y) \hookrightarrow L^{3p/2}(Y)$. Hence, we have a multiplication map $B^{1/p,p}(Y) \times B^{1/p,p}(Y) \hookrightarrow L^{3p/4}(Y)$. Next, the projection $\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}$ onto $\mathcal{K}^{s,p}_{(B_0,\Psi_0)}$ extends to a bounded map on $L^{3p/4}\mathcal{T}$ since (B_0,Ψ_0) is sufficiently regular (see the proof of Lemma 4.7). Finally when we apply the inverse Hessian, we get an element of $H^{1,3p/4}(Y)$ (Proposition 3.20 generalizes to Sobolev spaces, see Remark 4.17). Since we have an embedding $H^{1,3p/4}(Y) \hookrightarrow B^{1-1/p,p}(Y) \subseteq B^{1/p,p}(Y)$, this shows that $Q_{(B_0,\Psi_0)}$ is bounded on $B^{1/p,p}(Y)$. In the other direction, suppose $x \in L^3\mathcal{T}_{(B_0,\Psi_0)}$ and $F_{(B_0,\Psi_0)}(x) \in \mathcal{T}^{1/p,p}_{(B_0,\Psi_0)}$. In this situation, we have $Q_{(B_0,\Psi_0)}(x) \in H^{1,3/2}\mathcal{T}_{(B_0,\Psi_0)}$, which embeds into $\mathcal{T}^{1/p,p}_{(B_0,\Psi_0)}$, and so it follows that $x \in \mathcal{T}^{1/p,p}_{(B_0,\Psi_0)}$. (For $t > 1/p$, we do not have $Q_{(B_0,\Psi_0)}(x) \in \mathcal{T}^{t,p}_{(B_0,\Psi_0)}$, which is why we need $q > 3$ so that we have room to elliptic bootstrap.)

All the steps in Theorem 4.8 follow through as before to prove the corollary for $t = 1/p$. The arithmetic for the $H^{t,p}$ spaces yields the same result. □

4.1 Boundary Values of the Space of Monopoles

Define the space of tangential boundary values of monopoles

$$\mathcal{L}^{s-1/p,p}(Y,\mathfrak{s}) = r_\Sigma(\mathfrak{M}^{s-1/p,p}(Y,\mathfrak{s})). \tag{4.30}$$

By (4.10), we also have

$$\mathcal{L}^{s-1/p,p}(Y,\mathfrak{s}) = r_\Sigma(\mathcal{M}^{s,p}(Y,\mathfrak{s})). \tag{4.31}$$

With Y and $\mathfrak{s}$ fixed and satisfying (4.1), we simply write $\mathcal{L}^{s-1/p,p} = \mathcal{L}^{s-1/p,p}(Y,\mathfrak{s})$.

We know that $\mathcal{M}^{s,p}$ is a manifold for $s > \max(3/p, 1/2)$ by Theorem 4.2. Under further restrictions on s, we will see that $\mathcal{L}^{s-1/p,p}$ is also a manifold and the restriction map $r_\Sigma : \mathcal{M}^{s-1/p,p} \to \mathcal{L}^{s-1/p,p}$ is a covering map with fiber $\mathcal{G}_{h,\partial}(Y)$, which, as defined in (3.11), is the gauge group of harmonic gauge transformations which restrict to the identity on Σ. Furthermore, this covering map implies that the chart maps for $\mathcal{M}^{s,p}$ push forward under r_Σ to chart maps for the manifold $\mathcal{L}^{s-1/p,p}$. Consequently, the smoothing properties of the chart maps for $\mathcal{M}^{s,p}$ in Theorem 4.8 induce chart maps for $\mathcal{L}^{s-1/p,p}$ that have similar smoothing properties.

First, we establish several important lemmas.

Lemma 4.10 *For $s > \max(3/p, 1/2)$, $r_\Sigma : \mathcal{M}^{s,p} \to \mathfrak{C}^{s-1/p,p}(\Sigma)$ is an immersion.*

Proof This is just Corollary 3.18(ii). □

The following important lemma allows us to control the norm of a monopole on Y in terms of the norm of its restriction on Σ.

Lemma 4.11 *Let $s - 1/p > 1/2$ or $s \geq 1$ if $p = 2$. Then there exists a continuous function $\mu_{s,p} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $(B,\Psi) \in \mathcal{M}^{s,p}$, we can find a gauge transformation*

$g \in \mathcal{G}_{h,\partial}(Y)$ such that

$$\|g^*(B - B_{\mathrm{ref}}, \Psi)\|_{B^{s,p}(Y)} \le \mu_{s,p}\left(\|r_\Sigma(B - B_{\mathrm{ref}}, \Psi)\|_{B^{s-1/p,p}(\Sigma)}\right). \tag{4.32}$$

Proof For the moment, assume $(B, \Psi) \in \mathfrak{C}(Y)$ is any smooth configuration. Define the following quantities

$$\mathcal{E}^{\mathrm{an}}(B,\Psi) = \frac{1}{4}\int_Y |F_{B^t}|^2 + \int_Y |\nabla_B \Psi|^2 + \frac{1}{4}\int_Y (|\Psi|^2 + (s/2))^2 - \int_Y \frac{s^2}{16} \tag{4.33}$$

$$\mathcal{E}^{\mathrm{top}}(B,\Psi) = -\int_\Sigma (D_B^\partial \Psi, \Psi) + \int (H/2)|\Psi|^2. \tag{4.34}$$

Here s is the scalar curvature of Y, H is the mean curvature of Σ, and D_B^∂ is the boundary Dirac operator

$$(D_B^\partial \Psi)|_\Sigma = (\rho(\nu)^{-1} D_B \Psi)|_\Sigma - (\nabla_{B,\nu}\Psi)|_\Sigma + (H/2)\Psi|_\Sigma,$$

where ∇_B is the spin^c covariant derivative determined by B. Thus, D_B^∂ only involves differentiation along the directions tangential to Σ.

If we view (B, Ψ) as a time-independent configuration for the four-dimensional Seiberg-Witten equations (see the discussion before Theorem 17.3), then the above quantities are the analytic and topological energy of (B, Ψ), respectively, as defined in [21]. According to [21, Proposition 4.5.2], we have the energy identity

$$\mathcal{E}^{\mathrm{an}}(B,\Psi) = \mathcal{E}^{\mathrm{top}}(B,\Psi) + \|SW_3(B,\Psi)\|^2_{L^2(Y)}. \tag{4.35}$$

Observe that

$$\begin{aligned}\mathcal{E}^{\mathrm{top}}(B,\Psi) &\le C\left(\|\Psi\|^2_{B^{1/2,2}(\Sigma)} + \|\Psi\|^2_{L^3(\Sigma)}\|(B - B_{\mathrm{ref}})|_\Sigma\|_{L^3(\Sigma)} + \|\Psi\|^2_{L^2(\Sigma)}\right)\\ &\le C'\|r_\Sigma(B - B_{\mathrm{ref}}, \Psi)\|^3_{B^{1/2,2}(\Sigma)}\end{aligned}$$

for some constants C, C' independent of (B, Ψ). Here we used the embedding $B^{1/2,2}(\Sigma) \hookrightarrow L^3(\Sigma)$.

In what follows, we will use $x \lesssim y$ to denote $x \le Cy$ for some constant C that does not depend on the configuration (B, Ψ). Now consider a smooth solution of the three-dimensional Seiberg-Witten equations. Then we have $SW_3(B, \Psi) = 0$, and so it follows that

$$\mathcal{E}^{\mathrm{an}}(B,\Psi) \lesssim \|r_\Sigma(B - B_{\mathrm{ref}}, \Psi)\|^3_{B^{1/2,2}(\Sigma)}. \tag{4.36}$$

From this and the definition of $\mathcal{E}^{\mathrm{an}}(B, \Psi)$, we get the a priori bound

$$\|\Psi\|_{L^4(Y)} \lesssim 1 + \|r_\Sigma(B - B_{\mathrm{ref}}, \Psi)\|^3_{B^{1/2,2}(\Sigma)}. \tag{4.37}$$

If $(B, \Psi) \in \mathfrak{M}^{s,p}$ is not smooth, we can approximate (B, Ψ) by smooth configurations by Theorem 4.8(iii). We have $r_\Sigma(B - B_{\mathrm{ref}}, \Psi) \in \mathcal{T}_\Sigma^{s-1/p,p} \hookrightarrow \mathcal{T}_\Sigma^{s-1/p-\epsilon,2}$ for any $\epsilon > 0$ by Theorem 13.17. Since $s > 1/2 + 1/p$, we can choose ϵ so that $s - 1/p - \epsilon > 1/2$. Thus, we have uniform control over the $B^{1/2,2}(\Sigma)$ norm of the tangential boundary values of an approximating sequence to (B, Ψ). Thus, taking the limit, we see that (4.37) also holds for $(B, \Psi) \in \mathfrak{M}^{s,p}(Y)$.

Our remaining task is to use the a priori control (4.37) and the elliptic estimates for the Seiberg-Witten equations in Coulomb gauge to bootstrap our way to the estimate (4.32). By Corollary 15.22, we have the following elliptic estimate on 1-forms b:

$$\|b\|_{B^{t,q}(Y)} \lesssim \|db\|_{B^{t-1,q}(Y)} + \|d^*b\|_{B^{t-1,q}(Y)} + \|b|_\Sigma\|_{B^{t-1/q,q}(\Sigma)} + \|b^h\|_{B^{t-1,q}(Y)} \tag{4.38}$$

where b^h is the orthogonal projection of b onto the finite dimensional space

$$H^1(Y, \Sigma; i\mathbb{R}) \cong \{a \in \Omega^1(Y; i\mathbb{R}) : da = d^*a = 0, a|_\Sigma = 0\}. \tag{4.39}$$

Here $t > 1/q$ and $q \geq 2$.

Now let $(B, \Psi) \in \mathcal{M}^{s,p}$ be any configuration. Since it is in in the Coulomb slice determined by B_{ref}, then equation (4.38) implies

$$\|(B-B_{\mathrm{ref}})\|_{B^{t,q}(Y)} \lesssim \|F_{B^t}-F_{B^t_{\mathrm{ref}}}\|_{B^{t-1,q}(Y)}+\|r_\Sigma(B-B_{\mathrm{ref}})\|_{B^{t-1/q,q}(\Sigma)}+\|(B-B_{\mathrm{ref}})^h\|_{B^{t-1,q}(Y)} \tag{4.40}$$

where t, q will be chosen later. Since Dirichlet boundary conditions are overdetermined for the smooth Dirac operator $D_{B_{\mathrm{ref}}}$, we have the elliptic estimate

$$\|\Psi\|_{B^{t,q}(Y)} \lesssim \|D_{B_{\mathrm{ref}}}\Psi\|_{B^{t-1,q}(Y)} + \|\Psi\|_{B^{t-1/q,q}(\Sigma)}. \tag{4.41}$$

There exists an absolute constant C such that for any configuration (B_0, Ψ_0), we can find a gauge transformation $g \in \mathcal{G}_{h,\partial}(Y)$ such that $g^*(B_0, \Psi_0)$ satisfies $\|(g^*(B_0-B_{\mathrm{ref}}))^h\|_{B^{t-1,q}(Y)} \leq C$, since the quotient of $H^1(Y, \Sigma; i\mathbb{R})$ by the lattice $\mathcal{G}_{h,\partial}(Y)$ is a torus. To keep notation simple, redefine (B, Ψ) by such a gauge transformation. Such a gauge transformation preserves containment in $\mathcal{M}^{s,p}$ since the monopole equations are gauge invariant and the Coulomb-slice is preserved by $\mathcal{G}_{h,\partial}(Y)$. So using the bound $\|(B - B_{\mathrm{ref}})^h\|_{B^{t-1,q}(Y)} \leq C$ and the identity $SW_3(B, \Psi) = 0$, the bounds (4.40) and (4.41) become

$$\|(B - B_{\mathrm{ref}})\|_{B^{t,q}(Y)} \lesssim \|\Psi^2\|_{B^{t-1,q}(Y)} + \|(B - B_{\mathrm{ref}})|_\Sigma\|_{B^{t-1/q,q}(\Sigma)} + 1. \tag{4.42}$$

$$\|\Psi\|_{B^{t,q}(Y)} \lesssim \|\rho(B - B_{\mathrm{ref}})\Psi\|_{B^{t-1,q}(Y)} + \|\Psi\|_{B^{t-1/q,q}(\Sigma)}. \tag{4.43}$$

We will use these estimates, bootstrapping in t and q and using the a priori control (4.37), to get the estimate (4.32).

Let us first consider the case $p = 2$ and $s \geq 1$. Letting $t = 1$ and $q = 2$, (4.42) and (4.37) yield

$$\begin{aligned}\|B - B_{\mathrm{ref}}\|_{B^{1,2}(Y)} &\lesssim 1 + \|\Psi\|_{L^4(Y)} + \|(B - B_{\mathrm{ref}})|_\Sigma\|_{B^{1/2,2}(\Sigma)} \\ &\lesssim 1 + \|r_\Sigma(B - B_{\mathrm{ref}}, \Psi)\|^3_{B^{1/2,2}(\Sigma)}.\end{aligned}$$

This yields control over $\|B - B_{\mathrm{ref}}\|_{L^4(Y)}$ since we have the embedding $B^{1,2}(Y) \hookrightarrow L^6(Y)$.

Using this estimate in (4.43) with $t = 1$, $q = 2$, to control $\rho(B - B_{\text{ref}})$, we have

$$\begin{aligned}\|\Psi\|_{B^{1,2}(Y)} &\lesssim \|B - B_{\text{ref}}\|_{L^4(Y)}\|\Psi\|_{L^4(Y)} + \|\Psi\|_{B^{1/2,2}(\Sigma)}\\ &\lesssim 1 + \|r_\Sigma(B - B_{\text{ref}}, \Psi)\|^6_{B^{1/2,2}(\Sigma)}.\end{aligned}$$

This proves the estimate for $s = 1$. The estimate (4.32) for $s \geq 1$ now follows from boostrapping the elliptic estimates (4.42) and (4.43) in t. Indeed, once we gain control over $\|(B, \Psi)\|_{B^{t,q}(Y)}$, we can control the quadratic terms $\|\Psi^2\|_{B^{t'-1,q}(Y)}$ and $\|\rho(B - B_{\text{ref}})\Psi\|_{B^{t'-1,q}(Y)}$ for some $t' > t$ as long as $t' \leq s$. After finitely many steps of bootstrapping, we get (4.32), where the function $\mu_{s,p}$ can be computed explicitly if desired.

For $p > 2$ and $s \geq 1$, we use the imbedding $\mathfrak{C}^{s-1/p,p}(\Sigma) \hookrightarrow \mathfrak{C}^{s-1/p-\epsilon,2}(\Sigma)$, for any $\epsilon > 0$. From the previous case, we find that we can control $\|(B, \Psi)\|_{B^{s,2}(Y)}$ in terms of $\|(B, \Psi)\|_{B^{s-1/2,2}(\Sigma)}$. Since $B^{s,2}(Y) \subseteq B^{1,2}(Y) \hookrightarrow L^6(Y)$, the quadratic terms in (4.42) and (4.43) lie in $L^3(Y)$. Since we have the embedding $L^3(Y) \subset B^{0,q}(Y)$, where $q = \max(3, p)$, we can repeat the bootstrapping process (in t) as in the previous case to the desired estimate (4.32) for any $s \geq 1$ and $p \leq 3$. Suppose $p > 3$. Then with $q = 3$ in the previous step, we have established (4.42) and (4.43) with $t = 1$ and $q = 3$. Since $B^{1,3}(Y) \hookrightarrow L^q(Y)$ for any $q < \infty$, we have control of the quadratic terms of (4.42) and (4.43) in L^p for any $p < \infty$. Thus, we have the estimate (4.42) and (4.43) for $t = 1$ and $q = p$, since $L^p(Y) \subseteq B^{0,p}(Y)$. We can then bootstrap in t to the estimate (4.32) for any $s \geq 1$ and $p < \infty$. Thus, we have taken care of the case $s \geq 1$ and all $p \geq 2$.

Finally, suppose $s < 1$ and $p > 2$. We employ the same strategy of bootstrapping in q until we get to p. Since $s - 1/p > 1/2$, we have $B^{s-1/p,p}(\Sigma) \hookrightarrow B^{1/2,2}(\Sigma)$ and so we have control of $\|(B - B_{\text{ref}}, \Psi)\|_{B^{1,2}(Y)}$ and $\|(B - B_{\text{ref}}, \Psi)\|_{L^6(Y)}$ in terms of $\|(B - B_{\text{ref}})|_\Sigma\|_{B^{1/2,2}(\Sigma)}$. Let $1/2 < t = s < 1$ and $q = \min(3, p)$ in (4.42) and (4.43). We have control of the quadratic terms on the right-hand side since $L^3(Y) \subset B^{0,q}(Y) \subset B^{s-1,q}(Y)$, since $s - 1 \leq 0$. Thus, we have the control (4.32) for $p = q$. If $p \leq 3$, we are done. Else $p > 3$ and we bootstrap in q. Indeed, starting with $q_1 = 3$, we have a map $B^{s,q_i}(Y) \times B^{s,q_i}(Y) \to B^{2s-3/q_i,q_i}(Y) \hookrightarrow L^{q_{i+1}}(Y) \subseteq B^{s-1,q_{i+1}}(Y)$, where $q_{i+1} = q_i/(2(1 - sq_i)) > q_i$. Using (4.42) and (4.43), we thus bootstrap to the estimate (4.32) with $p = q_{i+1}$ from the estimate (4.32) with $p = q_i$. The q_i keep increasing until after finitely many steps, we get to the desired p, thereby proving (4.32). $\square$

The next lemma tells us that any two monopoles which have the same restriction to Σ are gauge equivalent on Y.

Lemma 4.12 *Let $s > \max(3/p, 1/2)$. If $(B_1, \Psi_1), (B_2, \Psi_2) \in \mathfrak{M}^{s,p}$ and $r_\Sigma(B_1, \Psi_1) = r_\Sigma(B_2, \Psi_2)$, then (B_1, Ψ_1) and (B_2, Ψ_2) are gauge equivalent on $\mathfrak{C}^{s,p}(Y)$.*

Proof Because of (4.10), without loss of generality, we can suppose $(B_1, \Psi_1), (B_2, \Psi_2) \in \mathcal{M}^{s,p}$. There are two cases to consider. In the first case, one and hence both the configurations are reducible. Indeed, if say (B_1, Ψ_1) is reducible, then $\Psi_2|_\Sigma = \Psi_1|_\Sigma = 0$. Since $D_{B_2}\Psi_2 = 0$, by unique continuation for D_{B_2}, we have $\Psi_2 \equiv 0$ so that (B_2, Ψ_2) is also reducible. In this reducible case, then B_1 and B_2 are both flat connections and so by (4.1), we must have $H^1(Y, \Sigma) = 0$. Since $d(B_1 - B_2) = d^*(B_1 - B_2) = 0$ and by hypothesis $(B_1 - B_2)|_\Sigma = 0$, we must then have $B_1 - B_2 = 0$ since $H^1(Y, \Sigma) = 0$. So in this case,

B_1 and B_2 are in fact equal. In the second case, neither configuration is reducible. In this case, consider the 4-manifold $S^1 \times Y$ and regard (B_1, Ψ_1) and (B_2, Ψ_2) as time-independent solutions to the Seiberg-Witten equations on $S^1 \times Y$. We now apply Theorem 17.3. □

Piecing the previous lemmas together, we can now prove the rest of our main theorem concerning the monopole spaces:

Theorem 4.13 *Let $s > \max(3/p, 1/2 + 1/p)$. Then $\mathcal{L}^{s-1/p,p}$ is a closed Lagrangian submanifold of $\mathfrak{C}^{s-1/p,p}(\Sigma)$. Furthermore, the maps*

$$r_\Sigma : \mathfrak{M}^{s,p} \to \mathcal{L}^{s-1/p,p} \tag{4.44}$$

$$r_\Sigma : \mathcal{M}^{s,p} \to \mathcal{L}^{s-1/p,p} \tag{4.45}$$

are a submersion and a covering map respectively, where the fiber of the latter is the lattice $\mathcal{G}_{h,\partial}(Y) \cong H^1(Y, \Sigma)$.

Proof By (4.10), it suffices to consider the map (4.45). By Lemma 4.10, the map $r_\Sigma : \mathcal{M}^{s,p} \to \mathfrak{C}^{s-1/p,p}(\Sigma)$ is an immersion, hence a local embedding. The previous lemma implies that (4.45) is injective modulo $G := \mathcal{G}_{h,\partial}(Y)$, since the gauge transformations which restrict to the identity and preserve Coulomb gauge are precisely those gauge transformations in G. Moreover, G acts freely on $\mathcal{M}^{s,p}(Y)$ by assumption (4.1), since when there are reducible solutions, we have $G = 1$.

It remains to show that $r_\Sigma : \mathcal{M}^{s,p}/G \to \mathfrak{C}^{s-1/p,p}(\Sigma)$ is an embedding onto its image. Let $(B_i, \Psi_i) \in \mathcal{M}^{s,p}$, $i \geq 1$, be such that $r_\Sigma(B_i, \Psi_i) \to r(B_0, \Psi_0)$ in $\mathfrak{C}^{s-1/p,p}(\Sigma)$ as $i \to \infty$. We want to show that given any subsequence of the (B_i, Ψ_i), there exists a further subsequence convergent to an element of the G orbit of (B_0, Ψ_0). This, combined with the fact that (4.45) is a local embedding will imply that (4.45) is a global embedding, modulo the covering transformations G. Indeed, the local embedding property tells us that there exists a open neighborhood $V_{(B_0,\Psi_0)} \ni (B_0, \Psi_0)$ of $\mathcal{M}^{s,p}$ such that $r_\Sigma : V_{(B_0,\Psi_0)} \to \mathfrak{C}^{s-1/p,p}(\Sigma)$ is an embedding onto its image, and moreover, $r_\Sigma(g^*V_{(B_0,\Psi_0)}) = r_\Sigma(V_{(B_0,\Psi_0)})$ for all $g \in G$. Proving the above convergence result shows that given a sufficiently small neighborhood U of $r_\Sigma(B_0, \Psi_0)$ in $\mathfrak{C}^{s-1/p,p}(\Sigma)$, then $U \cap \mathcal{L}^{s-1/p,p}$ is contained in the image of any one of the embeddings $r_\Sigma : g^*V_{(B_0,\Psi_0)} \to \mathfrak{C}^{s-1/p,p}(\Sigma)$, $g \in G$. Otherwise, we could find a subsequence $(B_{i'}, \Phi_{i'})$ of the (B_i, Ψ_i) such that $r_\Sigma(B_{i'}, \Phi_{i'}) \to r_\Sigma(B_0, \Psi_0)$ but the $(B_{i'}, \Phi_{i'})$ lie outside all the $g^*V_{(B_0,\Psi_0)}$, a contradiction.

Without further ado then, by Lemma 4.11, we know we can find gauge transformations $g_i \in G$ such that $g_i^*(B_i, \Psi_i)$ is uniformly bounded in $B^{s,p}(Y)$, since $r_\Sigma(B_i, \Psi_i) \to r_\Sigma(B_0, \Psi_0)$ is uniformly bounded. For notational simplicity, redefine the (B_i, Ψ_i) by these gauge transformations. Thus, since the (B_i, Ψ_i) are bounded in $B^{s,p}(Y)$, any subsequence contains a weakly convergent subsequence. Let $(B_\infty, \Psi_\infty) \in \mathcal{M}^{s,p}(Y)$ be a weak limit of some subsequence $(B_{i'}, \Psi_{i'})$. We have $r_\Sigma(B_\infty, \Psi_\infty) = r_\Sigma(B_0, \Psi_0)$, and so (B_∞, Ψ_∞) and (B_0, Ψ_0) are gauge equivalent by an element of G. If we can show that that $(B_{i'}, \Psi_{i'}) \to (B_\infty, \Psi_\infty)$ strongly in $B^{s,p}(Y)$, then we will be done. Due to the compact embedding $B^{s,p}(Y) \hookrightarrow B^{t,p}(Y)$, for $t < s$, we have $(B_i, \Psi_i) \to (B_\infty, \Psi_\infty)$ strongly in the topology $B^{s-\epsilon,p}(Y)$, $\epsilon > 0$. If we can boostrap this to strong convergence in $B^{s,p}(Y)$, we will be done. To show this, we use the ellipticity of $\widetilde{\mathcal{H}}_{(B_\infty,\Psi_\infty)}$. Let $(b_i, \psi_i) = (B_i - B_\infty, \Psi_i - \Psi_\infty)$. We have the elliptic estimate

$$\|(b_i,\psi_i)\|_{B^{s,p}(Y)} \lesssim \|\widetilde{\mathcal{H}}_{(B_\infty,\Psi_\infty)}(b_i,\psi_i)\|_{B^{s-1,p}(Y)} + \|r_\Sigma(b_i,\psi_i)\|_{B^{s-1/p,p}(\Sigma)}. \tag{4.46}$$

This follows because $\widetilde{\mathcal{H}}_{(B_\infty,\Psi_\infty)}$ is elliptic and the boundary term controls the kernel of $\widetilde{\mathcal{H}}_{(B_\infty,\Psi_\infty)}|_{\mathcal{C}^{s,p}}$ by Corollary 3.18(ii). The last term of (4.46) tends to zero and for the first term, we have

$$\widetilde{\mathcal{H}}_{(B_0,\Psi_0)}(b_i,\psi_i) = (b_i,\psi_i)\#(b_i,\psi_i) \tag{4.47}$$

from (4.15), since $(B_i,\Psi_i),(B_\infty,\Psi_\infty) \in \mathcal{M}^{s,p}$. We have a continuous multiplication map $B^{s-\epsilon,p}(Y)\times B^{s-\epsilon,p}(Y) \to B^{s-\epsilon,p}(Y) \subset \mathcal{B}^{s-1,p}(Y)$ for $s-\epsilon > 3/p$. Since $(B_i,\Psi_i)\to(B_0,\Psi_0)$ strongly in $B^{s-\epsilon,p}(Y)$, we have that (4.47) goes to zero in $B^{s-1,p}(Y)$, which means $(b_{i'},\psi_{i'})$ goes to zero in $B^{s,p}(Y)$ by (4.46). Thus, $(B_i,\Psi_i)\to(B_\infty,\Psi_\infty)$ strongly in $B^{s,p}(Y)$.

It now follows that $r_\Sigma : \mathfrak{M}^{s,p}(Y) \to \mathfrak{C}^{s-1/p,p}(\Sigma)$ is a covering map onto a embedded submanifold, where the fiber of the cover is G. Moreover, the proof we just gave also shows that $\mathcal{L}^{s-1/p,p}$ is a closed submanifold, since if $r_\Sigma(B_i,\Psi_i)$ is a convergent sequence, it is convergent to $r_\Sigma(B_\infty,\Psi_\infty)$ for some $(B_\infty,\Psi_\infty)\in\mathcal{M}^{s-1/p,p}$. Finally, Theorem 3.13(i) shows that $\mathcal{L}^{s-1/p,p}$ is Lagrangian submanifold of $\mathfrak{C}^{s-1/p,p}(\Sigma)$, since its tangent space at any point is a Lagrangian subspace of $\mathcal{T}_\Sigma^{s-1/p,p}$. □

Since $r_\Sigma : \mathcal{M}^{s,p} \to \mathcal{L}^{s-1/p,p}$ is a covering, the chart maps on $\mathcal{M}^{s,p}$ push forward and induce chart maps on $\mathcal{L}^{s-1/p,p}$. Indeed, at the tangent space level, we already know we have isomorphisms $r_\Sigma : T_{(B_0,\Psi_0)}\mathcal{M}^{s,p} \to T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$ and $P_{(B_0,\Psi_0)} : T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p} \to T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$ inverse to one another, where recall $P_{(B_0,\Psi_0)}$ is the Poisson operator given by Theorem 3.13. Because $\mathcal{M}^{s,p}$ is locally a graph over $T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$, then $\mathcal{L}^{s-1/p,p}$ is locally a graph over $T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$. To analyze this properly, we also want to "push foward" the local straightening map $F_{(B_0,\Psi_0)}$ for $\mathcal{M}^{s,p}$ at a configuration (B_0,Ψ_0), defined in Lemma 4.6, to obtain a local straightening map $F_{\Sigma,(B_0,\Psi_0)}$ for $\mathcal{L}^{s-1/p,p}$ at $r_\Sigma(B_0,\Psi_0)$.

Lemma 4.14 *Let $s > \max(3/p, 1/2+1/p)$ and let $(B_0,\Psi_0)\in\mathcal{M}^{s,p}$. Define the spaces*

$$X_\Sigma = \mathcal{T}_\Sigma^{s-1/p,p}, \qquad X_{\Sigma,0} = T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}, \qquad X_{\Sigma,1} = J_\Sigma X_{\Sigma,0}.$$

We have $X_\Sigma = X_{\Sigma,0}\oplus X_{\Sigma,1}$ and we can define the smooth map

$$\begin{aligned} F_{\Sigma,(B_0,\Psi_0)} &: V_\Sigma \to X_{\Sigma,0}\oplus X_{\Sigma,1} \\ x = (x_0,x_1) &\mapsto (x_0, x_1 - r_\Sigma E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}x_0)), \end{aligned} \tag{4.48}$$

where $V_\Sigma \subset X_\Sigma$ is an open subset containing 0 and $E^1_{(B_0,\Psi_0)}$ is defined as in Theorem 4.8. For any $\max(1/2,2/p) < s' \le s-1/p$, we can take V_Σ to contain a $B^{s',p}(\Sigma)$ ball, i.e., there exists a $\delta>0$, depending on $r_\Sigma(B_0,\Psi_0)$, s', and p, such that

$$V_\Sigma \supseteq \{x\in X_\Sigma : \|x\|_{B^{s',p}(\Sigma)} < \delta\}.$$

Moreover, we have the following:

(i) We have $F_{\Sigma,(B_0,\Psi_0)}(0)=0$ and $D_0F_{\Sigma,(B_0,\Psi_0)} = \mathrm{id}$. For V_Σ sufficiently small, $F_{\Sigma,(B_0,\Psi_0)}$

is a local straightening map for $\mathcal{L}^{s-1/p,p}$ at $r_\Sigma(B_0,\Psi_0)$ within the neighorhood V_Σ.

(ii) We can choose δ uniformly for $r_\Sigma(B_0,\Psi_0)$ in a sufficiently small $B^{s',p}(\Sigma)$ neighborhood of any configuration in $\mathcal{L}^{s-1/p,p}$.

Proof We have $F_{\Sigma,(B_0,\Psi_0)}(0)=0$ since $E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}(0))=E^1_{(B_0,\Psi_0)}(0)=0$, and $D_0F_{\Sigma,(B_0,\Psi_0)}=\mathrm{id}$ since $D_0E^1_{(B_0,\Psi_0)}=0$ by Theorem 4.8. Moreover, we see that $F_{\Sigma,(B_0,\Psi_0)}$ can be defined on a $B^{s',p}(\Sigma)$ ball containing $0\in X_\Sigma$. Indeed, $P_{(B_0,\Psi_0)}$ maps such a ball into a $B^{s'+1/p,p}(Y)$ ball inside $T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$, we have the embedding $B^{s'+1/p,p}(Y)\hookrightarrow L^\infty(Y)$ by our choice of s', and the domain of $E^1_{(B_0,\Psi_0)}$ contains an $L^\infty(Y)$ ball by Theorem 4.8. Take V_Σ to be such a $B^{s',p}(\Sigma)$ ball.

It now follows from $\mathcal{L}^{s-1/p,p}\subset\mathcal{L}^{s',p}$ and the fact that $r_\Sigma:\mathcal{M}^{s'+1/p,p}\to\mathcal{L}^{s',p}$ is a covering map onto a globally embedded submanifold (by Theorem 4.13) that $F_{\Sigma,(B_0,\Psi_0)}$ is a local straightening map for $\mathcal{L}^{s-1/p,p}$ within a $B^{s',p}(\Sigma)$ neighborhood of $0\in X_\Sigma$. (Shrinking V_Σ if necessary, let this neighborhood be V_Σ.) In more detail, if $x\in V_\Sigma$ and $F_{\Sigma,(B_0,\Psi_0)}(x)\in X_{\Sigma,0}$, then

$$r_\Sigma(B_0,\Psi_0)+x=r_\Sigma(B_0,\Psi_0)+(x_0,r_\Sigma E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}x_0)),\tag{4.49}$$

which means that

$$r_\Sigma(B_0,\Psi_0)+x=r_\Sigma\Big((B_0,\Psi_0)+P_{(B_0,\Psi_0)}x_0+E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}x_0)\Big),\tag{4.50}$$

where $(B_0,\Psi_0)+P_{(B_0,\Psi_0)}x_0+E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}x_0)\in\mathcal{M}^{s,p}$ by Theorem 4.8. Thus, $r_\Sigma(B_0,\Psi_0)+x\in r_\Sigma(\mathcal{M}^{s,p})=\mathcal{L}^{s-1/p,p}$. Conversely, if $x\in V_\Sigma$ is such that $r_{\Sigma(B_0,\Psi_0)}+x\in\mathcal{L}^{s-1/p,p}$, then (having chosen V_Σ small enough) we must have

$$r_\Sigma(B_0,\Psi_0)+x=r_\Sigma\Big((B_0,\Psi_0)+x_0'+E^1_{(B_0,\Psi_0)}(x_0')\Big).\tag{4.51}$$

for some $x_0'\in T_{(B_0,\Psi_0)}\mathcal{M}^{s'+1/p,p}$ since $\mathcal{L}^{s-1/p,p}\subset\mathcal{L}^{s',p}$ and $r_\Sigma:\mathcal{M}^{s'+1/p,p}\to\mathcal{L}^{s',p}$ is a local diffeomorphism from a neighborhood of $(B_0,\Psi_0)\in\mathcal{M}^{s'+1/p,p}$ onto a neighborhood of $r_\Sigma(B_0,\Psi_0)\in\mathcal{L}^{s',p}$. Since $P_{(B_0,\Psi_0)}:T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s',p}\to T_{(B_0,\Psi_0)}\mathcal{M}^{s'+1/p,p}$ is an isomorphism, then $x_0'=P_{(B_0,\Psi_0)}x_0$ for some $x_0\in T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s',p}$ and so (4.50), hence (4.49) must hold. By definition of $F_{\Sigma,(B_0,\Psi_0)}$, which extends to a well-defined map on the $B^{s'+1/p,p}(\Sigma)$ topology, (4.49) implies $x_0=F_{\Sigma,(B_0,\Psi_0)}(x)$. But $F_{\Sigma,(B_0,\Psi_0)}$ acting on a neighborhood of 0 in $\mathcal{T}^{s',p}_\Sigma$ preserves the $B^{s-1/p,p}(\Sigma)$ topology, so $x_0\in T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$ since $x\in\mathcal{T}^{s-1/p,p}_\Sigma$. Thus, $F_{\Sigma,(B_0,\Psi_0)}(x)\in X_{\Sigma,0}$. Moreover, both $F_{\Sigma,(B_0,\Psi_0)}$ and $F^{-1}_{\Sigma,(B_0,\Psi_0)}$ are invertible when restricted to V_Σ, since the inverse $F^{-1}_{\Sigma,(B_0,\Psi_0)}$ is simply given by

$$F^{-1}_{\Sigma,(B_0,\Psi_0)}(x)=(x_0,x_1+r_\Sigma E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}x_0)).\tag{4.52}$$

Altogether, this shows that $F_{\Sigma,(B_0,\Psi_0)}$ is a local straightening map for $\mathcal{L}^{s-1/p,p}$ within V_Σ.

(ii) This follows from the uniformity statement of Theorem 4.8(ii) and the continuous dependence of $P_{(B_0,\Psi_0)}:\mathcal{T}^{s',p}_\Sigma\to\mathcal{T}^{s'+1/p,p}\hookrightarrow L^\infty\mathcal{T}$ with respect to (B_0,Ψ_0) (see Theorem 3.13(iv)). Here, we use the fact that if $r_\Sigma(B_0,\Psi_0)\in\mathcal{L}^{s-1/p,p}$ varies continuously in a small $B^{s',p}(\Sigma)$ neighborhood, then one can choose $(B_0,\Psi_0)\in\mathcal{M}^{s,p}$ continuously in a small $B^{s'+1/p,p}(Y)$ neighborhood, since $r_\Sigma:\mathcal{M}^{s'+1/p,p}\to\mathcal{L}^{s',p}$ and $r_\Sigma:\mathcal{M}^{s,p}\to\mathcal{L}^{s-1/p,p}$ are covers. □

With the above lemma, we have local straightening maps for our Banach manifold $\mathcal{L}^{s-1/p,p}$. Then from Theorem 4.8 and the general framework of Section 20, we have the following theorem for the local chart maps for $\mathcal{L}^{s-1/p,p}$.

Theorem 4.15 *Let $s>\max(3/p,1/2+1/p)$.*

(i) Let $(B_0,\Psi_0)\in\mathcal{M}^{s,p}$. Then there exists a neighborhood $U\subset T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$ of 0 and a map $E^1_{r_\Sigma(B_0,\Psi_0)}:U\to X_{\Sigma,1}$, where $X_{\Sigma,1}$ is as in Lemma 4.14, such that the map

$$\begin{aligned} E_{r_\Sigma(B_0,\Psi_0)}:U&\to\mathcal{L}^{s-1/p,p}\\ x&\mapsto r_\Sigma(B_0,\Psi_0)+x+E^1_{r_\Sigma(B_0,\Psi_0)}(x) \end{aligned}\tag{4.53}$$

is a diffeomorphism of U onto a neighborhood of $r_\Sigma(B_0,\Psi_0)$ in $\mathcal{L}^{s-1/p,p}$. Furthermore, the map $E^1_{r_\Sigma(B_0,\Psi_0)}$ smooths by one derivative, i.e. $E^1_{r_\Sigma(B_0,\Psi_0)}(x)\in\mathcal{T}_\Sigma^{s+1-1/p,p}$ for all $x\in U$.

(ii) For any $\max(1/2,2/p)<s'\le s-1/p$, we can choose U such that both U and $E_{r_\Sigma(B_0,\Psi_0)}(U)$ contain $B^{s',p}(U)$ neighborhoods, i.e., there exists a $\delta>0$, depending on $r_\Sigma(B_0,\Psi_0)$, s', and p, such that

$$\begin{aligned} U&\supseteq\{x\in T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}:\|x\|_{B^{s',p}(\Sigma)}<\delta\}\\ E_{r_\Sigma(B_0,\Psi_0)}(U)&\supseteq\{(B,\Psi)\in\mathcal{L}^{s-1/p,p}:\|(B,\Psi)-r_\Sigma(B_0,\Psi_0)\|_{B^{s',p}(\Sigma)}<\delta\}, \end{aligned}$$

The constant δ can be chosen uniformly in $r_\Sigma(B_0,\Psi_0)$, for all $r_\Sigma(B_0,\Psi_0)$ in a sufficiently small $B^{s-1/p,p}(Y)$ neighborhood of any configuration in $\mathcal{L}^{s-1/p,p}$.

(iii) Smooth configurations are dense in $\mathcal{L}^{s-1/p,p}$.

Proof (i) As in (4.27), the chart map $E_{r_\Sigma(B_0,\Psi_0)}$ is determined by restricting $F^{-1}_{\Sigma,(B_0,\Psi_0)}$, the inverse of the local straightening map $F_{\Sigma,(B_0,\Psi_0)}$, to a neighborhood of 0 in the tangent space $T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$. Thus, we have

$$E_{r_\Sigma(B_0,\Psi_0)}(x)=r_\Sigma(B_0,\Psi_0)+F^{-1}_{\Sigma,(B_0,\Psi_0)}(x),\qquad x\in U:=F_{\Sigma,(B_0,\Psi_0)}(V_\Sigma)\cap T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p},\tag{4.54}$$

where V_Σ is defined as in Lemma 4.14. The expression for $F^{-1}_{\Sigma,(B_0,\Psi_0)}$ is given by (4.52). Thus, (4.54) and the definition of $E^1_{(B_0,\Psi_0)}$ in (4.53) yields

$$E^1_{r_\Sigma(B_0,\Psi_0)}(x)=r_\Sigma E^1_{(B_0,\Psi_0)}(P_{(B_0,\Psi_0)}x)\tag{4.55}$$

The mapping properties of $E^1_{r_\Sigma(B_0,\Psi_0)}$ now follow from Theorem 4.8.

(ii) This is a direct consequence of Lemma 4.14(ii).

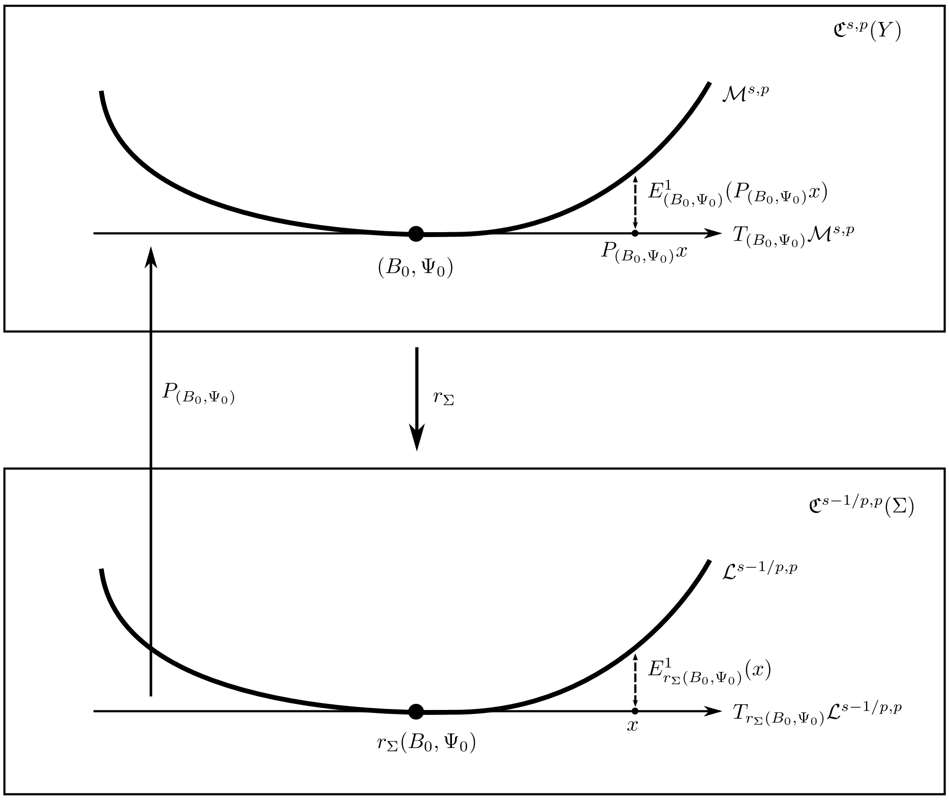

Figure I-2: A chart map for $\mathcal{M}^{s,p}$ at (B_0, Φ_0) induces a chart map for $\mathcal{L}^{s-1/p,p}$ at $r_\Sigma(B_0, \Phi_0)$.

(iii) This just follows from $r_\Sigma : \mathcal{M}^{s,p} \to \mathcal{L}^{s-1/p,p}$ being a cover and the density of smooth configurations in $\mathcal{M}^{s,p}$ by Theorem 4.8. □

Corollary 4.16 *Suppose* $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$.

(i) If U *is a sufficiently small* $L^p(\Sigma)$ *neighborhood of* 0 *in* $L^pT_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$*, then* $E_{r_\Sigma(B_0,\Psi_0)}$ *extends to a bounded map*

$$E_{r_\Sigma(B_0,\Psi_0)} : U \to L^p\mathfrak{C}(\Sigma). \tag{4.56}$$

The map (4.56) is a diffeomorphism onto its image and hence $E_{r_\Sigma(B_0,\Psi_0)}(U)$ *is an* L^p *submanifold of* $L^p\mathfrak{C}(\Sigma)$ *contained in* $L^p\mathcal{L}$*.*

(ii) The $L^p(\Sigma)$ *topology above can be replaced with* $B^{t,p}(\Sigma)$ *for any* $0 \le t \le s - 1/p$ *and* $H^{t,p}(\Sigma)$ *for any* $0 \le t \le s - 1/p$*.*

Proof We use Corollary 4.9 to show that $E^1_{r_\Sigma(B_0,\Psi_0)}$ is bounded on the $L^p(\Sigma)$ topology. We only prove the lowest regularity case $s = 0$, since the other cases are similar (and more easily handled). We have the inclusion $L^p \subset B^{0,p}$ since $p \ge 2$. By Theorem 3.13, the Poisson operator $P_{(B_0,\Psi_0)}$ maps $B^{0,p}(\Sigma)$ to $B^{1/p,p}(Y)$ for $s \ge 0$ since (B_0, Ψ_0) is sufficiently regular. In the proof of Corollary 4.9, we showed that $E^1_{(B_0,\Psi_0)}$ maps $B^{1/p,p}(Y)$ to $H^{1,3p/4}(Y)$. Hence when we apply r_Σ, we find altogether from (4.55) that $E^1_{r_\Sigma(B_0,\Psi_0)}(x)$

belongs to $B^{1-4/3p,3p/4}(\Sigma) \hookrightarrow L^p(\Sigma)$. Thus, $E^1_{(B_0,\Psi_0)}$ is bounded on $L^p(\Sigma)$ and $B^{0,p}(\Sigma)$. In the above calculation, we implicitly used the fact that $H^{1/p,p}(Y) \hookrightarrow L^3(Y)$, so that $P_{(B_0,\Psi_0)}$ maps a small $L^p(\Sigma)$ ball into the domain of $E^1_{(B_0,\Psi_0)}$, which contains an $L^3(Y)$ ball by Corollary 4.9. □

Remark 4.17 We have mentioned before that since our analysis works on a variety of function spaces, it is merely a matter of convenience that we worked primarily with Besov spaces on Y. In the above corollary and elsewhere, we see how the usual L^p spaces can be employed as well. Corollary 4.16 will be significant in Part III, since we will need to consider, locally, L^p closures of $\mathcal{L}$. We conclude by noting that every instance in which the $B^{s,p}(Y)$ topology is used in Part I, the topology $H^{s,p}(Y)$ may be used instead. These spaces, known as the Bessel potential spaces, are defined in Section 13. For s a nonnegative integer and $1 < p < \infty$, we have $H^{s,p}(Y) = W^{s,p}(Y)$, the Sobolev space of functions having s derivatives belonging to $L^p(Y)$. When $p = 2$, $H^{s,2}(Y) = B^{s,2}(Y)$ for all s. Furthermore, the spaces $H^{s,p}$ and $B^{s,p}$ are "close" to each other in the sense that $H^{s_1,p}(Y) \subseteq B^{s_2,p}(Y) \subseteq H^{s_3,p}(Y)$ for all $s_1 > s_2 > s_3$. Moreover, one sees that all the foundational analysis in Section 15 applies equally to Bessel potential and Besov spaces.

We should note that two particular places where it is important that Sobolev spaces may be used in addition to Besov spaces are Lemma 3.4 and Proposition 3.20. Indeed, their proofs rely only on function space arithmetic and elliptic estimates arising from elliptic boundary value problems. For both of these, Sobolev spaces can be used all the same, and so we can replace every occurrence of the $B^{\bullet,\bullet}(Y)$ topology with the $H^{\bullet,\bullet}(Y)$ topology in Lemma 3.4 and Proposition 3.20. One can now check that the statements of all our lemmas and theorems concerning Besov spaces on Y also hold for their Sobolev counterparts.

For the purposes of Part III, it is also important that we can replace Besov spaces on Σ with Sobolev spaces on Σ as well, but with some care, since the space of boundary values of a Sobolev space is still a Besov space. We already saw how to do this in Corollary 4.16. We should note that for the Calderon projection $P^+_{(B_0,\Psi_0)}$ in Theorem 3.13, where $(B_0, \Psi_0) \in B^{s,p}(Y)$, one also has that

$$P^+_{(B_0,\Psi_0)} : H^{t-1/p,p}\mathcal{T}_\Sigma \to H^{t-1/p,p}\mathcal{T}_\Sigma, \qquad t < s+1. \tag{4.57}$$

is bounded. This follows from the fact that π^+ is bounded on $H^{t-1/p,p}\mathcal{T}_\Sigma$, as it is a pseudodifferential operator, and

$$(P^+_{(B,\Psi)} - \pi^+) : H^{t-1/p,p}\mathcal{T}_\Sigma \subseteq \mathcal{T}_\Sigma^{t-1/p,p} \to \mathcal{T}_\Sigma^{\min(s-1/p+1,\, t-1/p+1),p} \subset H^{t-1/p,p}\mathcal{T}_\Sigma$$

by Theorem 3.13(iv) and Theorem 13.17.

From the above remark, the Sobolev version of our main theorem, with the $H^{s,p}(Y)$ topology replaced with the $B^{s,p}(Y)$ topology, holds. In fact, one can see from this that the Besov monopole space $\mathcal{M}^{s,p}$ is actually equal to the Sobolev monopole space $H^{s,p}\mathcal{M}$.[21]

[21] From the density of smooth configurations, it suffices to show that the tangent space to $H^{s,p}\mathcal{M}$ and $\mathcal{M}^{s,p}$ at a smooth monopole (B, Ψ) are both equal. However, this follows from the fact that the kernel of an elliptic operator (in our case, the operator $\widetilde{\mathcal{H}}_{(B,\Psi)}$) in the $H^{s,p}$ and $B^{s,p}$ topologies are equivalent. This follows from the results of Section 15.3, which shows that these spaces are isomorphic (modulo a finite dimensional

Finally, let us remark that the proof of the main corollary of Part I easily follows from the work we have done.

Proof of Main Corollary: For every coclosed 1-form η, the zero set of $SW_3(B,\Phi) = (\eta, 0)$ is gauge-invariant. Thus, all the methods of Section 3.3 apply to the linearization of the monopole spaces associated to the perturbed equations. Next, we still have the transversality result Lemma 4.1 so long as we modify the assumption (4.1) to $c_1(\mathfrak{s}) \neq 2[*\eta]$ or $H^1(Y,\Sigma) = 0$. The energy estimates in Lemma 4.11 still hold in the perturbed case since we still have (4.35) and the uniform bound $\|SW_3(B,\Psi)\|_{L^2} = \|\eta\|_{L^2}$. Finally, the unique continuation results from Section 17 still apply, since we always apply these results to the difference of solutions to the perturbed Seiberg-Witten equations, and the equation satisfied by the difference is independent of η. Thus, all our methods and hence results carry through in the perturbed case.

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subspace) to their space of boundary values, which is a fixed subspace of $B^{s-1/p,p}$ on the boundary.

Part II

The Seiberg-Witten Equations on a 3-manifold with Cylindrical Ends

In Part I, we studied the Seiberg-Witten equations on a compact 3-manifold with boundary. We now consider the case when our 3-manifold Y is allowed to have cylindrical ends. Our main task in to study the case when $Y = [0, \infty) \times \Sigma$ is a semi-infinite cylinder. From this, the general case can be understood by decomposing Y into a union of semi-infinite ends and a compact manifold with boundary and then piecing together the results on each of these parts.

A summary of the results we obtain is as follows. First, on a semi-infinite cylinder, we study the space of finite energy solutions to the Seiberg-Witten equations. We relate this space to the moduli space of vortices on Σ. Specifically, in Theorem 7.2, we show that the moduli space of all finite energy monopoles on Y is a Hilbert manifold weakly homotopy equivalent to the space of those monopoles with small energy. Furthermore, the restriction map to the boundary data on $\{0\} \times \Sigma$ sends this moduli space to a coisotropic submanifold of the the symplectic quotient of the boundary configuration space. Here, we have a moment map arising from the symplectic action of the gauge group $\mathcal{G}(\Sigma)$ on the boundary (see Proposition 5.1). In Theorem 7.6, we prove that for sufficiently small energy ϵ_0, the moduli space of monopoles on $Y = [0, \infty) \times \Sigma$ with energy less than ϵ_0, suitably topologized, is a Hilbert manifold diffeomorphic to a Hilbert bundle over the moduli space of vortices on Σ. Moreover, the space of all finite energy monopoles is weakly homotopy equivalent to those with small energy, and hence topologically, we see that these monopole spaces have a very simple structure.

Our explicit description of this boundary value space of monopoles is particularly convenient, because we have now have a way of obtaining Lagrangians in the boundary configuration space whose topology, rather than being completely mysterious, is equivalent to that of a finite dimensional manifold. Via Theorem 7.7, if we pick a Lagrangian submanifold $\mathscr{L}$ of the finite dimensional symplectic moduli space of vortices, the boundary values on $\{0\} \times \Sigma$ of those monopoles on Y that converge modulo gauge to vortices in $\mathscr{L}$ produce for us a Lagrangian weakly homotopy equivalent to a Hilbert bundle over $\mathscr{L}$. This is in contrast to the results of Part I, where the homotopy type of the Lagrangians we obtain from a compact 3-manifold is unclear how to determine.

Finally, in Theorem 8.2, we piece together the results here and those of Part I to understand in what sense the moduli space of Seiberg-Witten monopoles on a 3-manifold with cylindrical ends yields a Lagrangian correspondence between vortex moduli spaces on the ends. We are able to show that, after a suitable perturbation of the equations, the boundary values of our monopole spaces yield immersed Lagrangians. Here, one must be a bit careful in the choice of perturbations.

One application of our work is that it provides the supporting analysis for Donaldson's topological quantum field theoretic approach to the Seiberg-Witten invariants in [10][1]. There, Donaldson provides a beautiful, albeit formal proof, of certain topological results, including the Meng-Taubes formula for the Seiberg-Witten invariants. The proof rests on assuming the Seiberg-Witten equations on 3-manifolds with cylindrical ends provide Lagrangian correspondences (at least on the linear level) between the vortex moduli spaces of the Riemann surfaces at the end. Indeed, the TQFT aspect of this setup is that monopole moduli spaces on cobordisms provide the morphisms in the theory. Our main theorem, Theorem 8.2, and the discussion afterwards, shows that the analysis here can be used to supply the missing details in [10][2].

5 The Seiberg-Witten Flow on Σ

Let $Y = [0,\infty) \times \Sigma$, where Σ is a connected Riemann surface. Let $t \in [0,\infty)$ be the time-variable. Since, by convention, Σ is co-oriented by the outward unit normal, it follows that as an oriented manifold $Y = [0,\infty)_{\mathrm{opp}} \times \Sigma$, where $[0,\infty)_{\mathrm{opp}}$ denotes $[0,\infty)$ with the opposite orientation, i.e. it is oriented by the vector field $-\partial_t$. We will always assume that Y is oriented this way from now on, though we write $Y = [0,\infty) \times \Sigma$ for short.

Let (B,Ψ) be a smooth solution to $SW_3(B,\Psi) = 0$ on Y. We will always take the spin^c structure on Y to be pulled back from a spin^c structure on Σ, and by abuse of notation, we denote both of these spin^c structures by $\mathfrak{s}$. With respect to this product structure, we can write the equations $SW_3(B,\Psi) = 0$ in a rather explicit fashion. Recall that every Kahler manifold has a canonical spin^c structure (see [29]). For a Riemann surface Σ, the spinor bundle associated to this canonical spin^c structure is isomorphic to $K_\Sigma^{1/2} \oplus K_\Sigma^{-1/2}$, where K_Σ is the canonical bundle of Σ. Moreover, on Σ, a spin^c structure is uniquely determined by its determinant line bundle L, and the corresponding spinor bundle it determines is isomorphic to

$$\mathcal{S}_\Sigma \cong (K_\Sigma \otimes L)^{1/2} \oplus (K_\Sigma^{-1} \otimes L)^{1/2}. \tag{5.1}$$

Let $\pi_\Sigma : [0,\infty) \times \Sigma \to \Sigma$ denote the natural projection. From the above, every spin^c structure on $[0,\infty) \times \Sigma$ pulled back from a spin^c structure on Σ via π_Σ determines a spinor bundle on $[0,\infty) \times \Sigma$ isomorphic to

$$\pi_\Sigma^*(K_\Sigma \otimes L)^{1/2} \oplus \pi_\Sigma^*(K_\Sigma^{-1} \otimes L)^{1/2}. \tag{5.2}$$

Since $T^*([0,\infty)\times\Sigma) \cong T^*[0,\infty) \oplus T^*\Sigma$, we can always choose our Clifford multiplication ρ on

[1] In this regard, the author would like to especially thank Tim Perutz for this reference and subsequent helpful discussions.

[2] In light of this, our work also completes the proof of other results that rely upon results in [10]. For example, in [27], our work completes the proof of Theorem 4.1, since while [10] computes a quantity signified by the left-hand side of (11), [10] does not rigorously prove that it is equal to the mysterious right-hand side.

$[0,\infty)\times\Sigma$ to be such that ρ factors through the direct sum decomposition of $T^*([0,\infty)\times\Sigma)$. From this, we can choose ρ so that

$$\rho(-\partial_t) \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with respect to the decomposition (5.2). Using local holomorphic coordinates $z = x + iy$ on Σ, we can decompose a 1-form on Y into its dt, dz, and $d\bar{z}$ components. Given a spin^c connection B on L, let $F_{x,y}dx \wedge dy + F_{x,t}dx \wedge dt + F_{y,t}dy \wedge dt$ denote the local coordinate representation of F_B, the curvature of B. Then the equation $*F_B + \rho^{-1}(\Psi\Psi_0^*) = 0$ appearing in $SW_3(B,\Psi) = 0$ can be written explicitly as (see [30])[3]:

$$\left(F_{x,y} + \frac{i}{2}(|\Psi_+|^2 - |\Psi_-|^2)\right) dt = 0 \tag{5.3}$$

$$-\frac{1}{2}(F_{y,t} - iF_{x,t})d\bar{z} + \bar{\Psi}_+\Psi_- = 0 \tag{5.4}$$

$$-\frac{1}{2}(F_{y,t} + iF_{x,t})dz + \Psi_+\bar{\Psi}_- = 0 \tag{5.5}$$

Here, $\Psi = (\Psi_+, \Psi_-)$ is the decomposition of Ψ with respect to (5.2), so that $\bar{\Psi}_+\Psi_-$ and $\Psi_+\bar{\Psi}_-$ are well-defined elements of $\pi_\Sigma^*\mathcal{K}_\Sigma^\mp$, respectively. Observe that the last equation above is just the complex conjugate of the second. Moreover, the Dirac equation $D_B\Psi = 0$ becomes

$$\begin{pmatrix} i\nabla_{B,-\partial_t} & \sqrt{2}\bar{\partial}^*_{B|_\Sigma} \\ \sqrt{2}\partial_{B|_\Sigma} & -i\nabla_{B,-\partial_t} \end{pmatrix} \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} = 0, \tag{5.6}$$

where $\nabla_{B,-\partial_t}$ denotes the spin^c covariant derivative of B evaluated in the $-\partial_t$ direction. Thus, equations (5.3)–(5.6) yield for us the Seiberg-Witten equations on $[0,\infty)\times\Sigma$.

In the same way that the Seiberg-Witten equations on a product 4-manifold can be written as the downward flow of the Seiberg-Witten vector field on the slice 3-manifold (when the configuration in question is in temporal gauge), we want to reinterpret the Seiberg-Witten equations on $[0,\infty)\times\Sigma$ as a downward flow of a vector field on Σ. To do this, we can consider the oriented 4-manifold $S^1 \times [0,\infty)_{\mathrm{opp}} \times \Sigma$ and regard configurations on $[0,\infty)_{\mathrm{opp}} \times \Sigma$ as S^1 invariant. If we do this, and we place (B,Ψ) in temporal gauge, then since $S^1 \times [0,\infty)_{\mathrm{opp}} \times \Sigma = [0,\infty) \times S^1 \times \Sigma$ as oriented manifolds, we can regard (B,Ψ) as a downward flow for the Seiberg-Witten vector field on $S^1 \times \Sigma$:

$$\frac{d}{dt}(B,\Psi) = -SW_3^{S^1\times\Sigma}\left((B(t),\Psi(t))|_{S^1\times\Sigma}\right). \tag{5.7}$$

Here, $SW_3^{S^1\times\Sigma}$ denotes the gradient of the Chern-Simons-Dirac functional on $S^1\times\Sigma$. The Clifford multiplication $\tilde{\rho}$ on $S^1\times\Sigma$ is such that $\tilde{\rho}(\partial_\theta) = \rho(-\partial_t)$, where θ is the coordinate on S^1, and $\tilde{\rho}|_{T\Sigma} = \rho|_{T\Sigma}$. Now for any S^1 invariant configuration (B,Ψ) on $S^1\times\Sigma$, the

[3] Note that our sign conventions are that of [21], namely $\rho(-dtd\Sigma) = 1$, which is the opposite choice of sign in [30]). A switch in the two choices of Clifford multiplication is compensated by a bundle automorphism and complex conjugation of spin^c structures.

Chern-Simons-Dirac functional on on $S^1 \times \Sigma$ is given by

$$CSD(B,\Psi) = \frac{1}{2}\int_\Sigma \mathrm{Re}\,(\Psi, D_{B|_\Sigma}\Psi). \tag{5.8}$$

Here, the Chern-Simons term drops out since B has no S^1 dependence or S^1 component, the operator $D_{B|_\Sigma}$ is the induced Dirac operator on Σ, and the length of S^1 is normalized to unity.

Notation. We write C to denote a connection on Σ and Υ to denote a spinor on Σ, i.e. (C,Υ) is an element of the configuration space $\mathfrak{C}(\Sigma) = \mathcal{A}(\Sigma) \times \Gamma(\mathcal{S}_\Sigma)$ on Σ. This is because our general trend is to write (B,Ψ) for a 3-dimensional configuration and (A,Φ) for a 4-dimensional configuration in Part III. Likewise, we use c to denote a 1-form on Σ. On the other hand, within equations, we will also use C and c to denote various constants appearing in inequalities whose precise value is unimportant. To avoid confusion, we will also use "const" to denote the value of various constants, which may change from line to line.

In light of (5.8), we define the Chern-Simons-Dirac functional CSD^Σ on $\mathfrak{C}(\Sigma)$ by

$$CSD^\Sigma(C,\Upsilon) = \frac{1}{2}\int_\Sigma \mathrm{Re}\,(\Upsilon, D_C\Upsilon), \qquad (C,\Upsilon) \in \mathfrak{C}(\Sigma), \tag{5.9}$$

where $D_C : \Gamma(\mathcal{S}_\Sigma) \to \Gamma(\mathcal{S}_\Sigma)$ is the spin^c Dirac operator determined from C. The L^2-gradient of this functional is given by

$$SW_2(C,\Upsilon) := \nabla_{(C,\Upsilon)} CSD \tag{5.10}$$

$$= (\tilde{\rho}_\Sigma^{-1}(\Upsilon\Upsilon^*)_0, D_C\Upsilon), \tag{5.11}$$

where $\tilde{\rho}_\Sigma^{-1} : i\mathfrak{su}(\mathcal{S}_\Sigma) \to T\Sigma$ is the map $\tilde{\rho}^{-1} : i\mathfrak{su}(\mathcal{S}_\Sigma) \to T(S^1 \times \Sigma)$ composed with the projection onto the $T\Sigma$ factor. We can consider the formal downward gradient flow of CSD^Σ on $\mathfrak{C}(\Sigma)$

$$\frac{d}{dt}(C,\Upsilon) = -SW_2(C,\Upsilon). \tag{5.12}$$

Regarding the S^1 invariant configuration (B,Ψ) in (5.7) as a path of configurations $(B,\Psi) = (C(t),\Psi(t))$ in $\mathfrak{C}(\Sigma)$, we see that (5.12) differs from (5.7) from the fact that the term $SW_3^{S_1\times\Sigma}(B,\Psi)$ contains a $d\theta$ component, where θ denotes the coordinate on S^1. However, because B is S^1-invariant and therefore has no $d\theta$ component, equation (5.7) implies that the $d\theta$ component of $SW_3^{S_1\times\Sigma}(B,\Psi)$ is identically zero, i.e., we have a constraint.

Since $\tilde{\rho}(\partial_\theta) = \rho(-\partial_t)$, this constraint is none other than the equation (5.3). In light of this, given $(C,\Upsilon) \in \mathfrak{C}(\Sigma)$, define the map

$$\mu(C,\Upsilon) = \check{*}F_C + \frac{i}{2}(|\Upsilon_-|^2 - |\Upsilon_+|^2). \tag{5.13}$$

Here $\check{*}$ is the Hodge star on Σ and $\Upsilon = (\Upsilon_+,\Upsilon_-)$ is the decomposition of $\Upsilon \in \Gamma(\mathcal{S}_\Sigma)$ induced by the splitting (5.1).

Recall that the gauge group $\mathcal{G}(\Sigma) = \mathrm{Maps}(\Sigma, S^1)$ acts on $\mathfrak{C}(\Sigma)$ via

$$(C, \Upsilon) \mapsto g^*(C, \Upsilon) = (C - g^{-1}dg, g\Upsilon), \qquad g \in \mathcal{G}(\Sigma).$$

We have the following proposition concerning the map μ:

Proposition 5.1 *(i) The map $\mu : \mathfrak{C}(\Sigma) \to \Omega^0(\Sigma; i\mathbb{R})$ is the moment map for $\mathfrak{C}(\Sigma)$ associated to the gauge group action of $\mathcal{G}(\Sigma)$. Here, the symplectic form on $\mathfrak{C}(\Sigma)$ is given by*

$$\omega((a, \phi), (b, \psi)) = \int_\Sigma a \wedge b + \int_\Sigma \mathrm{Re}\,(\phi, \rho(-\partial_t)\psi), \qquad (a, \phi), (b, \psi) \in \mathcal{T}_\Sigma.$$

(ii) If $\Upsilon \not\equiv 0$, then $d_{(C,\Upsilon)}\mu : T_{(C,\Upsilon)}\mathfrak{C}(\Sigma) \to \Omega^0(\Sigma; i\mathbb{R})$ is surjective.

(iii) Let $Y = [0, \infty) \times \Sigma$. Then a configuration $(B, \Psi) = (C(t), \Upsilon(t))$ in temporal gauge on Y solves $SW_3(B, \Psi) = 0$ if and only if $(C(t), \Upsilon(t))$ solves

$$\frac{d}{dt}(C(t), \Upsilon(t)) = -SW_2(C(t), \Upsilon(t)) \tag{5.14}$$

$$\mu(C(t), \Upsilon(t)) = 0, \qquad t > 0. \tag{5.15}$$

(iv) For any $(C, \Upsilon) \in \mathfrak{C}(\Sigma)$, we have $d_{(C,\Upsilon)}\mu\Big(SW_2(C, \Upsilon)\Big) = 0$, that is, $SW_2(C, \Upsilon)$ is tangent to the level set $\mu^{-1}(0)$.

Proof (i) This is the statement that at every $(C, \Upsilon) \in \mathfrak{C}(\Sigma)$, every $(c, \upsilon) \in T_{(C,\Upsilon)}\mathfrak{C}(\Sigma)$, and every $\xi \in \Omega^0(\Sigma; i\mathbb{R})$, we have

$$\int_\Sigma d_{(C,\Upsilon)}\mu(c, \upsilon) \cdot \xi = \omega((-d\xi, \xi\Upsilon), (c, \upsilon)).$$

Verifying this is a straightforward computation.

(ii) The range of $\check{*}d : \Omega^1(\Sigma; i\mathbb{R}) \to \Omega^0(\Sigma; i\mathbb{R})$ consists of precisely those functions that integrate to zero on Σ. Suppose f is orthogonal to the image of $d_{(C,\Upsilon)}\mu$. If $\Upsilon \not\equiv 0$, then one can find $\upsilon \in \Gamma(\mathcal{S}_\Sigma)$ such that $d_{(C,\Upsilon)}\mu(0, \upsilon) = \int_\Sigma f$. It then follows that one can find a 1-form c such that $d_{(C,\Upsilon)}(c, \upsilon) = f$.

Statement (iii) follows from the preceding discussion. For (iv), we first observe that since μ is the moment map for the gauge group action on $\mathfrak{C}(\Sigma)$, then the kernel of its differential is the symplectic annihilator of the tangent space to the gauge orbit:

$$\begin{aligned}\ker d_{(C,\Upsilon)}\mu &= \{(-d\xi, \xi\Upsilon) : \xi \in \Omega^0(\Sigma; i\mathbb{R})\}^{\perp_\omega} \\ &= J\{(-d\xi, \xi\Upsilon) : \xi \in \Omega^0(\Sigma; i\mathbb{R})\}^{\perp}.\end{aligned}$$

Here,

$$J := J_\Sigma = (-\check{*}, \rho(\partial_t)) : \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma) \circlearrowleft$$

is the compatible complex structure for ω. Thus, to show $d_{(C,\Upsilon)}\mu(SW_2(C, \Upsilon)) = 0$, it suffices to show that $JSW_2(C, \Upsilon)$ is perpendicular to the tangent space to the gauge orbit of (C, Υ). For this, it suffices to show that $JSW_2(C, \Upsilon)$, like $SW_2(C, \Upsilon)$, is the gradient of a gauge-invariant functional. A simple computation shows that the gradient of the functional

$$(C, \Upsilon) \mapsto \frac{1}{2}\int_\Sigma (\Upsilon, \rho(\partial_t)D_C\Upsilon)$$

is $JSW_2(C, \Upsilon)$. Here, we use the fact that, by convention of our choice of Clifford multiplication, $\rho(-\partial_t)\rho(d\Sigma) = 1$ and so $\rho(\partial_t)\rho(c) = \rho(d\Sigma)\rho(c) = \rho(\check{*}c)$). □

The last statement of the above lemma implies that the restriction of the gradient of CSD^Σ to the level set $\mu^{-1}(0)$ is equal to the gradient of $CSD^\Sigma|_{\mu^{-1}(0)}$ (at points where $\mu^{-1}(0)$ is smooth). If $c_1(\mathfrak{s}) \neq 0$, then by (iii), $\mu^{-1}(0)$ is always a smooth submanifold of $\mathfrak{C}(\Sigma)$ since it contains no flat connections. Thus, we have the following corollary:

Corollary 5.2 *Suppose* $c_1(\mathfrak{s}) \neq 0$. *Then* $\mu^{-1}(0)$ *is a smooth submanifold of* $\mathfrak{C}(\Sigma)$ *and modulo gauge, solutions to* $SW_3(B, \Psi) = 0$ *on* Y *correspond to (formal) downward gradient flow lines of* $CSD^\Sigma|_{\mu^{-1}(0)}$.

Thus, the bulk of our analysis consists in understanding the gradient flow of $CSD^\Sigma|_{\mu^{-1}(0)}$.

5.1 The Vortex Equations

From now on, we always assume

$$c_1(\mathfrak{s}) \neq 0,$$

so that $\mu^{-1}(0)$ is a smooth manifold. Our first task is to understand the set of critical points of $CSD^\Sigma|_{\mu^{-1}(0)}$.

Let $d = \langle c_1(\mathfrak{s}), [\Sigma]\rangle$. By the above assumption, $d \neq 0$. In this case, we have the following facts. First, the critical points of $CSD^\Sigma|_{\mu^{-1}(0)}$ have an explicit description in terms of the space of vortices on Σ. Secondly, this critical set is Morse-Bott nondegenerate for $CSD^\Sigma|_{\mu^{-1}(0)}$. This is in contrast to the case $d = 0$, where although the critical set of $CSD^\Sigma|_{\mu^{-1}(0)}$ is just the space of flat connections on Σ, this set is in general Morse-Bott degenerate.[4]

For the sake of completeness, we describe in detail the correspondence between the critical set of $CSD^\Sigma|_{\mu^{-1}(0)}$ and the space of vortices, following [31]. Recall that the vortex equations on Σ are given by the following. Given a line bundle $E \to \Sigma$ over Σ of degree k, a Hermitian connection A on E, a section $\psi \in \Gamma(E)$, and a function $\tau \in \Omega^0(\Sigma; i\mathbb{R})$, the vortex equations are given by

$$\check{*}F_A - \frac{i|\psi|^2}{2} = \tau \tag{5.16}$$

$$\bar{\partial}_A\psi = 0. \tag{5.17}$$

Here, $\bar{\partial}_A : E \to K_\Sigma^{-1} \otimes E$ is the holomorphic structure on E determined by A. Observe that if $k > \frac{i\tau}{2\pi}$, there are no solutions to (5.16)–(5.17) by a simple application of the Chern-Weil theorem. When $0 < k < \int_\Sigma \frac{i\tau}{2\pi}$, then by [14], the moduli space of gauge equivalence classes of solutions $\mathcal{V}_{k,\tau}(\Sigma)$ to (5.16)–(5.17) can be naturally identified with the space of

[4] A flat connection C will be Morse-Bott degenerate precisely when $\ker D_C \neq 0$.

effective divisors of degree k on Σ, i.e. the k-fold symmetric product $\mathrm{Sym}^k(\Sigma)$ of Σ. This identification is given by mapping a solution (A, ψ) to the set of zeros of the (nontrivial) holomorphic section ψ. Because $\mathrm{Sym}^k(\Sigma)$ is independent of τ, we will often simply denote the moduli space of degree k vortices by $\mathcal{V}_k(\Sigma)$. Likewise, we will denote the space of *all* solutions to (5.16)–(5.17) by $\mathscr{V}_k(\Sigma)$.

Observe that if $k < 0$, one may instead consider the equations

$$\check{*}F_A + \frac{i|\psi|^2}{2} = -\tau \tag{5.18}$$

$$\partial_A \psi = 0, \tag{5.19}$$

which become equivalent to (5.16)–(5.17) via complex conjugation. We will call the equations (5.18)–(5.19) the anti-vortex equations. Thus, the moduli space $V_{k,\tau}(\Sigma)$ of solutions to (5.18)–(5.19) is nonempty for $\int_\Sigma -\frac{i\tau}{2\pi} < k < 0$ and can can be identified with $\mathcal{V}_{|k|,\tau}(\Sigma)$.

The equations that determine the critical points of CSD^Σ that belong to the zero set of the moment map are given by $\mu(C, \Upsilon) = 0$ and $SW_2(C, \Upsilon) = 0$. More explicitly, these equations are given by

$$\check{*}F_C + \frac{i}{2}(|\Upsilon_-|^2 - |\Upsilon_+|^2) = 0 \tag{5.20}$$

$$\Upsilon_+ \bar{\Upsilon}_- = 0 \tag{5.21}$$

$$\bar{\Upsilon}_+ \Upsilon_- = 0 \tag{5.22}$$

$$\bar{\partial}_C \Upsilon_+ = 0 \tag{5.23}$$

$$\bar{\partial}_C^* \Upsilon_- = 0. \tag{5.24}$$

We can now see the correspondence between equations (5.20)–(5.24) and the vortex equations (5.16)–(5.17). Equations (5.23)–(5.24) and unique continuation for Dirac operators implies that (5.21)–(5.22) forces

$$\Upsilon_+ \equiv 0 \qquad \text{or} \qquad \Upsilon_- \equiv 0.$$

Let g denote the genus of Σ. Pick a connection C_{g-1} on $K_X^{1/2}$ and define $\tau = \check{*}F_{C_{g-1}}$. We assume C_{g-1} is such that $\check{*}F_{C_{g-1}}$ is constant (and hence equal to $\frac{-2\pi i(g-1)}{\mathrm{Vol}(\Sigma)}$.) Let $\bar{C}_{g-1}$ denote the corresponding dual connection on $K_X^{-1/2}$. Then (5.20) is equivalent to each of the following equations

$$\check{*}F_{C\otimes C_{g-1}} + \frac{i}{2}(|\Upsilon_-|^2 - |\Upsilon_+|^2) = \tau \tag{5.25}$$

$$\check{*}F_{C\otimes \bar{C}_{g-1}} + \frac{i}{2}(|\Upsilon_-|^2 - |\Upsilon_+|^2) = -\tau \tag{5.26}$$

Let

$$k = g - 1 - \frac{|d|}{2}.$$

Using the constraints on k for when the vortex and anti-vortex moduli spaces $\mathcal{V}_k(\Sigma)$ and $V_{-|k|}(\Sigma)$ are nonempty, and the fact that a line bundle can have nontrivial holmorphic sections only if it has nonnegative degree, it is easy to see that the following situation holds:

Lemma 5.3 *With notation as above, we have the following:*

(i) *Suppose* $-2(g-1) < d < 0$. *Then the space of critical points of* $CSD^{\Sigma}|_{\mu^{-1}(0)}$ *corresponds precisely to the space of vortices* $\mathscr{V}_k(\Sigma)$ *under the correspondence* $(C, \Upsilon) \mapsto (C \otimes C_{g-1}, \Upsilon_+)$. *Here* Υ_- *vanishes identically.*

(ii) *Suppose* $0 < d < 2(g-1)$. *Then the space of critical points of* $CSD^{\Sigma}|_{\mu^{-1}(0)}$ *corresponds precisely to the space of anti-vortices* $\mathscr{V}_{-k}(\Sigma)$ *under the correspondence* $(C, \Upsilon) \mapsto (C \otimes \bar{C}_{g-1}, \Upsilon_-)$. *Here,* Υ_+ *vanishes identically.*

(iii) *If* $d = -2(g-1)$, *then the space of critical points of* $CSD^{\Sigma}|_{\mu^{-1}(0)}$ *is precisely the gauge orbit of a single configuration* $(\bar{C}_{g-1}, (\Upsilon_{g-1}, 0))$, *where* $\Upsilon_{g-1} \in \Omega^0(\Sigma; \mathbb{C})$ *satisfies* $|\Upsilon_{g-1}|^2 \equiv 2i{*}F_{C_{g-1}}$. *The analogous statement holds for* $d = 2(g-1)$.

(iv) *If* $|d| > 2(g-1)$, *then the set of critical points of* $CSD^{\Sigma}|_{\mu^{-1}(0)}$ *is empty.*

(v) *For all* $d \neq 0$, *the critical set of* $CSD^{\Sigma}|_{\mu^{-1}(0)}$ *is Morse-Bott nondegenerate.*

Proof Statements (i),(ii), and (iv) follow from the preceding analysis. For (iii), observe that a holomorphic vector bundle of degree zero has a nontrivial holomorphic section if and only if it is holomorphically trivial. Thus, modulo gauge, the connection C is zero and Υ_+ must be a constant section of a trivial line bundle. The norm constraint on Υ_+ now just follows from (5.25).

We need only prove (v). This amounts to showing the following. Given any configuration $(C_0, \Upsilon_0) \in \mathfrak{C}(\Sigma)$, let

$$\mathcal{H}_{2,(C_0,\Upsilon_0)} : \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma, \tag{5.27}$$

denote the *Hessian* of CSD^{Σ} at (C_0, Υ_0), which is the operator obtained by linearizing the map $SW_2 : \mathfrak{C}(\Sigma) \to \mathcal{T}_\Sigma$ at (C_0, Υ_0). If (C_0, Υ_0) is a vortex, we need to show that the restricted operator

$$\mathcal{H}_{2,(C_0,\Upsilon_0)} : T_{(C_0,\Upsilon_0)}\mu^{-1}(0) \to T_{(C_0,\Upsilon_0)}\mu^{-1}(0), \tag{5.28}$$

has kernel equal to precisely the tangent space to the space of vortices at (C_0, Υ_0). Without loss of generality, suppose $d < 0$. Then if we linearize the equations (5.20)–(5.24) at a vortex, then since $\Upsilon_- \equiv 0$ and Υ_+ vanishes only on a finite set of points, unique continuation shows that an element of the kernel of the linearized equations must have vanishing Ψ_- component. It follows that the only nontrivial equations we obtain are those obtained from linearizing (5.20) and (5.23), which yields for us precisely the linearization of the vortex equations. On the other hand, the space of vortices are cut out transversally by the vortex equations. (This is because the set $\{(A, \psi) : \bar{\partial}_A \psi = 0, \psi \not\equiv 0\}$ is a gauge-invariant Kahler submanifold of $\mathfrak{C}(\Sigma)$, the left-hand side of (5.16) is the moment map for this submanifold, and the gauge group acts freely on this submanifold.) It follows that the kernel of the map $\mathcal{H}_{2,(C_0,\Upsilon_0)}$ above is precisely the tangent space to the space of vortices. This finishes the proof of Morse-Bott nondegeneracy. □

By abuse of notation, given $d \neq 0$, we also write $\mathscr{V}_k(\Sigma)$ to denote the set of critical points of $CSD^\Sigma|_{\mu^{-1}(0)}$, which for $0 < k < g-1$, we may identify with the space of all degree k vortices on Σ by the above. We write $\mathcal{V}_k(\Sigma)$ to denote the quotient of $\mathscr{V}_k(\Sigma)$ by the gauge group, and it can be identified with $Sym^k(\Sigma)$ for all $0 \leq k < g-1$. For every k, note that the symplectic form on $\mathfrak{C}(\Sigma)//\mathcal{G}(\Sigma)$ restricts to a symplectic forms on the vortex moduli space $\mathcal{V}_k(\Sigma)$. We will refer to elements of either $\mathscr{V}_k(\Sigma)$ or $\mathcal{V}_k(\Sigma)$, for any k, simply as vortices. When Σ and k, fixed, we will often write $\mathscr{V}$ and $\mathcal{V}$ for brevity.

5.2 The Flow on a Slice

We now fix $0 \leq k < g-1$ throughout our discussion. In order to place ourselves in an elliptic situation and in a situation where we can apply Morse-Bott estimates to our configurations, we have to choose the right gauge for our equations. As it turns out, choosing a suitable gauge requires some careful setup. Our work here is modeled off that of [28], which studies the flow one obtains for the instanton equations on a cylindrical 4-manifold. To describe the gauge fixing procedure, we recall the basic gauge theoretic decompositions of the configuration space on Σ and its tangent spaces.

Our analysis proceeds mutatis mutandis as in Part I on a closed 3-manifold. Given a configuration $(C, \Upsilon) \in \mathfrak{C}(\Sigma)$, define

$$\mathcal{T}_{(C,\Upsilon)} = T_{(C,\Upsilon)}\mathfrak{C}(\Sigma) = \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma)$$

to be the tangent space to (C, Υ) of $\mathfrak{C}(\Sigma)$. If the basepoint is unimportant, we write $\mathcal{T}_\Sigma$ for any such tangent space. The infinitesimal action of the gauge group on $\mathfrak{C}(\Sigma)$ leads us to consider the following operators

$$\begin{aligned} \mathbf{d}_{(C,\Upsilon)} : \Omega^0(\Sigma; i\mathbb{R}) &\to \mathcal{T}_\Sigma \\ \xi &\mapsto (-d\xi, \xi\Upsilon) \\ \mathbf{d}^*_{(C,\Upsilon)} : \mathcal{T}_\Sigma &\to \Omega^0(\Sigma; i\mathbb{R}) \\ (c, v) &\mapsto -d^*c + i\mathrm{Re}\,(i\Upsilon, v), \end{aligned}$$

analogous to those considered in Section 3.2. From these operators, we obtain the following subspaces of $\mathcal{T}_{(C,\Upsilon)}$, which are the tangent space to the the gauge orbit through (C, Υ) and its orthogonal complement, respectively:

$$\begin{aligned} \mathcal{J}_{(C,\Upsilon)} &= \mathrm{im}\,\mathbf{d}_{(B,\Psi)} \\ \mathcal{K}_{(C,\Upsilon)} &= \ker \mathbf{d}^*_{(B,\Psi)}. \end{aligned}$$

As usual, we must consider the Banach space completion of the configuration spaces and the above vector spaces. Unlike Part I, where it was important that we work with low regularity Sobolev and Besov spaces to suit the needs of Part III, here such fine function space details are not of importance to us. Thus, we will consider only L^2 Sobolev spaces and we write $H^s(M)$ to denote the $H^{s,2}(M)$ topology on the manifold M, where the latter denotes the space of functions with s derivatives in $L^2(M)$, $s \in \mathbb{R}$. Otherwise, our notational conventions remain the same as in Part I.

Thus, we have $\mathfrak{C}^s(\Sigma)$, the $H^s(\Sigma)$ completion of the configuration space on Σ. Its tangent spaces are isomorphic to $\mathcal{T}^s_\Sigma$, the $H^s(\Sigma)$ completion of $\mathcal{T}_\Sigma$. For sufficiently regular (C,Υ), we obtain the following subspaces of $\mathcal{T}^s_\Sigma$:

$$\mathcal{J}^s_{(C,\Upsilon)} = \{(-d\xi, \xi\Upsilon) \in \mathcal{T}^s_{(C,\Upsilon)} : \xi \in H^{s+1}\Omega^0(\Sigma; i\mathbb{R})\}$$
$$\mathcal{K}^s_{(C,\Upsilon)} = \{(c,v) \in \mathcal{T}^s_{(C,\Upsilon)} : -d^*c + i\mathrm{Re}\,(i\Upsilon, v) = 0\}.$$

We have the following gauge-theoretic decompositions of the tangent space and configuration space:

Lemma 5.4 *Let $s > 0$. (i) Then for any $(C,\Upsilon) \in \mathfrak{C}^s(\Sigma)$, we have an L^2 orthogonal decomposition*

$$\mathcal{T}^s_{(C,\Upsilon)} = \mathcal{J}^s_{(C,\Upsilon)} \oplus \mathcal{K}^s_{(C,\Upsilon)}. \tag{5.29}$$

(ii) Define the slice

$$\mathfrak{S}^s_{(C_0,\Upsilon_0)} := (C_0,\Upsilon_0) + \mathcal{K}^s_{(C_0,\Upsilon_0)}$$

through (C_0,Υ_0) in $\mathcal{T}^s_{(C,\Upsilon)}$. There exists an $\epsilon > 0$ such that if $(C,\Upsilon) \in \mathfrak{C}^s(\Sigma)$ satisfies $\|(C,\Upsilon) - (C_0,\Upsilon_0)\|_{H^s(\Sigma)} < \epsilon$, then there exists a gauge transformation $g \in \mathcal{G}^{s+1}(\Sigma)$ such that $g^*(C,\Upsilon) \in \mathfrak{S}_{(C_0,\Upsilon_0)}$ and $\|g^*(C,\Upsilon) - (C_0,\Upsilon_0)\|_{H^s(\Sigma)} \le c_s\|(C,\Upsilon) - (C_0,\Upsilon_0)\|_{H^s(\Sigma)}$.

Proof (i) This follows from same analysis as in Lemma 3.4. (ii) This is an immediate consequence of the inverse function theorem and the fact that $\mathfrak{S}^s_{(C_0,\Upsilon_0)}$ is a local slice for the gauge action. □

For $s > 0$, define the quotient configuration space

$$\mathfrak{B}^s(\Sigma) = \mathfrak{C}^s(\Sigma)/\mathcal{G}^{s+1}(\Sigma).$$

Away from the reducible configurations, this quotient space is Hilbert manifold modeled on the above local slices (see [21]). The decomposition (5.29) allows us to define the complementary projections $\Pi_{\mathcal{J}^s_{(C,\Upsilon)}}$ and $\Pi_{\mathcal{K}^s_{(C,\Upsilon)}}$ of $\mathcal{T}^s_{(C,\Upsilon)}$ onto $\mathcal{J}^s_{(C,\Upsilon)}$ and $\mathcal{K}^s_{(C,\Upsilon)}$, respectively.

Let us return to the smooth setting for the time being. Denote the smooth quotient configuration space by

$$\mathfrak{B}(\Sigma) = \mathfrak{C}(\Sigma)/\mathcal{G}(\Sigma).$$

Our first task is rewrite the Seiberg-Witten equations on Y in a suitable gauge when the monopole in question is close to a vortex. This is so that we may exploit the Morse-Bott nature of the critical set, which we perform in the next section.

Notation. To simplify notation a bit, and to make it bear similarity with that of the standard reference [21], we introduce the following notation. We will write $\mathfrak{a}$ to denote a critical point of $CSD^\Sigma|_{\mu^{-1}(0)}$, i.e. a vortex. We will always assume $\mathfrak{a}$ is smooth, unless otherwise stated, since this can always be achieved via a gauge transformation. Given a configuration (B,Ψ) on $Y = [0,\infty)\times\Sigma$, we can write it as

$$(B,\Psi) = (C(t) + \beta(t)dt, \Upsilon(t))$$

where $(C(t),\Upsilon(t))$ is a path of configurations in $\mathfrak{C}(\Sigma)$ and $\beta(t)$ is a path in $\Omega^0(\Sigma; i\mathbb{R})$.

As shorthand, we will often write γ for the configuration (B,Ψ) and $\tilde{\gamma}(t)$ for the path $(C(t),\Upsilon(t))$. Given a vortex $\mathfrak{a}$, we write $\gamma_{\mathfrak{a}} \in \mathfrak{C}(Y)$ to denote the time-translation invariant path identically equal to $\mathfrak{a}$.

Given any vortex $\mathfrak{a} \in \mathscr{V}$, define

$$CSD^{\Sigma}_{\mathfrak{a}} = CSD^{\Sigma}|_{\mathfrak{S}_{\mathfrak{a}}} - CSD^{\Sigma}(\mathfrak{a}).$$

to be the restriction of CSD^{Σ} restricted to the slice through $\mathfrak{a}$, normalized by a constant for convenience. Note that CSD^{Σ} has a constant value on its critical set, since it is connected.

Since $SW_2(C,\Upsilon) = \nabla_{(C,\Upsilon)}CSD^{\Sigma}$ is the gradient of the gauge-invariant functional CSD^{Σ}, we know that $\nabla_{(C,\Upsilon)}CSD^{\Sigma}$ is orthogonal to $\mathcal{J}_{(C,\Upsilon)}$ and hence lies in $\mathcal{K}_{(C,\Upsilon)}$. On the other hand, if $(C,\Upsilon) \in \mathfrak{S}_{\mathfrak{a}}$, then the gradient of $CSD^{\Sigma}_{\mathfrak{a}}$ satisfies

$$\nabla_{(C,\Upsilon)}CSD^{\Sigma}_{\mathfrak{a}} \in \mathcal{K}_{\mathfrak{a}},$$

since a priori, this gradient must be tangent to the slice. For (C,Υ) close enough to $\mathfrak{a}$, then the space $J_{(C,\Upsilon)}$, which is automatically complementary to $\mathcal{K}_{(C,\Upsilon)}$, is also complementary to $\mathcal{K}_{\mathfrak{a}}$, and so $\nabla_{(C,\Upsilon)}CSD^{\Sigma}$ and $\nabla_{(C,\Upsilon)}CSD^{\Sigma}_{\mathfrak{a}}$ differ by an element of $\mathcal{J}_{(C,\Upsilon)}$. This suggests we introduce the following inner product structure on the tangent bundle of a neighborhood $\mathfrak{S}_{\mathfrak{a}}(\delta)$ of the slice (instead of the usual L^2 inner product). Namely, mimicking the construction in [28], consider the inner product

$$\langle x,y\rangle_{\mathfrak{a},(C,\Upsilon)} := \left(\Pi_{\mathcal{K}_{(C,\Upsilon)}}x, \Pi_{\mathcal{K}_{(C,\Upsilon)}}y\right)_{L^2(\Sigma)}, \qquad x,y \in T_{(C,\Upsilon)}\mathfrak{S}_{\mathfrak{a}}(\delta) \tag{5.30}$$

where $(\cdot,\cdot)_{L^2(\Sigma)}$ is the usual L^2 inner product on $\mathcal{T}_{\Sigma}$. As noted, for (C,Υ) sufficiently close to $\mathfrak{a}$, the map $\Pi_{\mathcal{K}_{(C,\Upsilon)}} : \mathcal{K}_{\mathfrak{a}} \to \mathcal{K}_{(C,\Upsilon)}$ is an isomorphism. Specifically, by the same analysis as in Remark 4.3, (C,Υ) in a small $H^{1/2}(\Sigma)$ ball U around $\mathfrak{a}$ is sufficient. Observe that the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{a},(C,\Upsilon)}$ naturally arises from pulling back the L^2 inner product on the irreducible part of the quotient configuration space $\mathfrak{C}(\Sigma)/\mathcal{G}(\Sigma)$.

Then if we endow the neighborhood U with the inner product (5.30), we can explicitly write $\nabla_{(C,\Upsilon)}CSD^{\Sigma}_{\mathfrak{a}}$ as follows. Let $\Pi_{\mathcal{K}_{\mathfrak{a}},\mathcal{J}_{(C,\Upsilon)}}$ denote the projection onto $\mathcal{K}_{\mathfrak{a}}$ through $\mathcal{J}_{(C,\Upsilon)}$, which exists for $(C,\Upsilon) \in U$ and U sufficiently small. Then

$$\nabla_{(C,\Upsilon)}CSD^{\Sigma}_{\mathfrak{a}} = \Pi_{\mathcal{K}_{\mathfrak{a}},\mathcal{J}_{(C,\Upsilon)}}SW_2(C,\Upsilon), \tag{5.31}$$

or in other words, there exists a well-defined map

$$\Theta_{\mathfrak{a}} : U \to \Omega^0(\Sigma; i\mathbb{R}) \tag{5.32}$$

such that

$$\nabla_{(C,\Upsilon)}CSD^{\Sigma}_{\mathfrak{a}} = SW_2(C,\Upsilon) - \mathbf{d}_{(C,\Upsilon)}\Theta_{\mathfrak{a}}(C,\Upsilon). \tag{5.33}$$

(The map $\Theta_{\mathfrak{a}}$ is well-defined since the operator $\mathbf{d}_{(C,\Upsilon)}$ is injective for (C,Υ) irreducible, which holds for U small.) The decomposition (5.33) is important because it relates the gradient vector field $\nabla_{(C,\Upsilon)}CSD^{\Sigma}_{\mathfrak{a}}$ to the vector field $SW_2(C,\Upsilon)$ by an infinitesimal action of the gauge group at the configuration (C,Υ). (Had we used the usual L^2 inner product, the analogous ansatz would have yielded an infinitesimal action of the gauge group at $\mathfrak{a}$ instead of the configuration (C,Υ) in question.)

Borrowing the terminology of [28], we introduce the following definition:

Definition 5.5 Fix $s \geq 1/2$.

(i) For any smooth vortex $\mathfrak{a}$, any open subset of $\mathfrak{S}_\mathfrak{a} \cap \mu^{-1}(0)$ of the form

$$U_\mathfrak{a}(\delta) := \{(C,\Upsilon) \in \mathfrak{S}^s_\mathfrak{a} \cap \mu^{-1}(0) : \|(C,\Upsilon) - \mathfrak{a}\|_{H^{1/2}} < \delta\}$$

for some small $\delta > 0$ is said to be a *coordinate patch* at $\mathfrak{a}$. We will often write $U_\mathfrak{a}$ to denote any such coordinate patch. We always assume that the (sufficiently small) coordinate patch $U_\mathfrak{a}$ is endowed with the inner product (5.30) on its tangent bundle.

(ii) Let I be a subinterval of $[0,\infty)$. Given a coordinate patch $U_\mathfrak{a}$ about a vortex $\mathfrak{a}$, we say that a configuration $\gamma \in \mathfrak{C}([0,\infty)\times\Sigma)$ is in *standard form* on $I \times \Sigma$ with respect to $U_\mathfrak{a}$ if $\check{\gamma}(t) \in U_\mathfrak{a}$ for all $t \in I$.

Our choice of defining $H^{1/2}$ open neighborhoods comes from our energy analysis of the next section. The value of s is immaterial for now and can be assumed as large as desired ($s \geq 2$ is sufficient). We will only need to consider Sobolev spaces of configurations in Section 7, where the usual functional analytic methods require we work with Hilbert space topologies.

The upshot of the above formalism is the following. Given a path of configurations $(C(t),\Upsilon(t))$ that is sufficiently near a vortex $\mathfrak{a}$ for all time t, we can gauge fix this path so that the new path lies in some neighborhood of $\mathfrak{a}$ in the slice $\mathfrak{S}_\mathfrak{a}$ for all time. The relevant situation is when this path of configurations is a monopole on $Y = [0,\infty)\times\Sigma$ in temporal gauge. When we perform such a gauge-fixing, two things happen. First, the resulting configuration γ determines a path $\check{\gamma}(t)$ in a coordinate patch $U_\mathfrak{a}$ (i.e., it is in standard form), since our monopole always determines a path in the zero set of the moment map by Corollary 5.2. Second, γ is no longer in temporal gauge. Nevertheless, the next lemma tells us that the resulting configuration γ is completely determined by the path $\check{\gamma}(t)$. Moreover, the path $\check{\gamma}(t)$ is simply a gradient flow line of $CSD^\Sigma_\mathfrak{a}$ restricted to $U_\mathfrak{a}$.

Lemma 5.6 *Let $\mathfrak{a}$ be a vortex and $U_\mathfrak{a}$ a coordinate patch. Let $\gamma = (C(t)+\beta(t)dt, \Upsilon(t))$ be a configuration on $[0,\infty)\times\Sigma$ in standard form on $[T_0,T_1]\times\Sigma$ with respect to $U_\mathfrak{a}$. Then γ satisfies $SW_3(\gamma) = 0$ if and only if*

$$\begin{aligned} \frac{d}{dt}\check{\gamma}(t) &= -\nabla_{\check{\gamma}(t)} CSD_\mathfrak{a} \\ \mathbf{d}^*_\mathfrak{a}(\check{\gamma}(t) - \mathfrak{a}) &= 0 \\ \mu(\check{\gamma}(t)) &= 0, \\ \beta(t) &= \Theta_\mathfrak{a}(\check{\gamma}(t)), \qquad T_0 < t < T_1. \end{aligned} \tag{5.34}$$

Proof The monopole equations $SW_3(\gamma) = 0$, as given by (5.6), are precisely

$$\begin{aligned} \frac{d}{dt}\check{\gamma}(t) &= -SW_2(\check{\gamma}(t)) + \mathbf{d}_{\gamma(t)}\beta(t) \\ \mu(\check{\gamma}(t)) &= 0. \end{aligned} \tag{5.35}$$

Thus, any solution to (5.34) yields a solution to (5.35). Conversely, suppose we have a solution γ to (5.35). Since γ is in standard form, it satisfies the second equation of (5.34), and taking a time-derivative of this equation, we obtain

$$\mathbf{d}_{\mathfrak{a}}^{*}\frac{d}{dt}\check{\gamma}(t) = 0.$$

The first equation now implies

$$-\mathbf{d}_{\mathfrak{a}}^{*}SW_2(\check{\gamma}(t)) + \mathbf{d}_{\mathfrak{a}}^{*}(\mathbf{d}_{\check{\gamma}(t)}\beta(t)) = 0.$$

From the definitions, this implies $\beta(t) = \Theta_{\mathfrak{a}}(\check{\gamma}(t))$. We now see that γ solves (5.34). □

We now use this lemma to study the asymptotic behavior of monopoles at infinity.

6 Asymptotic Convergence and Exponential Decay

Lemma 5.6 tells us that a solution to the Seiberg-Witten equations on $[0,\infty)\times\Sigma$ in standard form with respect to a small coordinate patch $U_{\mathfrak{a}}$ of a vortex $\mathfrak{a}$ satisfies the system of equations (5.34). These equations tell us that the solution γ is determined by the evolution of the path $\check{\gamma}(t)$ in $U_{\mathfrak{a}}$, since the normal component $\beta(t)dt$ is determined from $\check{\gamma}(t)$. The path $\check{\gamma}(t)$ is a downward gradient flow for the functional $CSD^{\Sigma}_{\mathfrak{a}}$ on the coordinate chart $U_{\mathfrak{a}}$, where this latter space has been endowed with the inner product (5.30). It is on a sufficiently small coordinate patch $U_{\mathfrak{a}}$ that we can apply standard Morse-Bott type estimates for the function $CSD^{\Sigma}_{\mathfrak{a}}$. These estimates imply that any trajectory $\check{\gamma}(t)$ that stays within $U_{\mathfrak{a}}$ for all time must converge exponentially fast to a critical point. Moreover, we can deduce that the $L^2(\Sigma)$ length of the path $\check{\gamma}(t)$ is bounded by the energy of the path, see (6.9). Here, the energy of a monopole γ is the quantity

$$\mathcal{E}(\gamma) = \int_0^\infty \|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)}dt. \tag{6.1}$$

Likewise we can define the energy $\mathcal{E}_I(\gamma)$ of a configuration on $I\times\Sigma$, for any interval $I = [t_1,t_2]$. On any such interval for which the energy is finite, the energy is equal to the drop in the Chern-Simons-Dirac functional on Σ:

$$CSD^{\Sigma}(\check{\gamma}(t_1)) - CSD^{\Sigma}(\check{\gamma}(t_2)) = \int_{t_1}^{t_2} \|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)}dt.$$

This is a simple consequence of the fact that a monopole on $I\times\Sigma$ is simply a downward gradient flow line of CSD^{Σ}.

Regarding a monopole (B,Ψ) on $I\times\Sigma$ as an S^1 invariant configuration on $S^1\times I\times\Sigma$, with I a compact interval, then we have the following energy identity (see [21]):

$$CSD^{\Sigma}(\check{\gamma}(t_1)) - CSD^{\Sigma}(\check{\gamma}(t_2)) = \int_{I\times\Sigma}\left(\frac{1}{4}|F_B|^2 + |\nabla_B\Psi|^2 + \frac{1}{4}\Big(|\Psi|^2 + (s/2)\Big)^2 - \frac{s^2}{16}\right)$$

where s is the scalar curvature of $I\times\Sigma$. Thus, modulo gauge, the energy of a monopole controls its H^1 norm on finite cylinders.

A key step in understanding the moduli space of finite energy monopoles is to show that if a monopole γ has small enough energy, then there is a vortex $\mathfrak{a}$ and a gauge transformation g on $[0,\infty)\times\Sigma$ such that $g^*\gamma$ determines a path that stays within some coordinate patch of $\mathfrak{a}$ for all time. In this way, one can see at an intuitive level what the moduli space of monopoles on $[0,\infty)\times\Sigma$ with small finite energy is. It is simply a neighborhood of the stable manifold to the space of vortices in the symplectic reduction $\mathfrak{C}(\Sigma)//\mathcal{G}(\Sigma)$. There is some analytic care that must be taken to establish this picture, however, since the coordinate patches we consider only contain $H^{1/2}(\Sigma)$ neighborhoods of a vortex $\mathfrak{a}$, whereas the important length estimate (6.9) is only an $L^2(\Sigma)$ bound. Nevertheless, it turns out that one can bootstrap the $L^2(\Sigma)$ convergence of the configuration to show that it converges in $H^s(\Sigma)$ exponentially fast to a vortex within a fixed coordinate chart, for all $s \geq 0$.

We begin with the following fundamental estimates for configurations with small energy. Given any I, we write $\mathscr{V}_I \subset \mathfrak{C}(I\times\Sigma)$ to denote the space of time translation invariant elements on $I\times\Sigma$ that belong to the space of vortices $\mathscr{V}$ for all time.

Lemma 6.1 *We have the following:*

(i) Given a bounded interval I, for every gauge invariant neighborhood V of $\mathscr{V}_I$ in $\mathfrak{C}^1(I\times\Sigma)$, there exists an $\epsilon>0$ such that if γ is any monopole on $I\times\Sigma$ satisfying the small energy condition $\int_I \|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)} < \epsilon$, then there exists a gauge transformation g such that $g^*\gamma \in V$.

(ii) For every gauge invariant neighborhood V_Σ of $\mathscr{V}$ in $\mathfrak{C}^1(\Sigma)$, there exists an $\epsilon>0$ such that if (C,Υ) is a configuration such that $\mu(C,\Upsilon)=0$ and $\|SW_2(C,\Upsilon)\|<\epsilon$, then there exists a gauge transformation g such that $g^*(C,\Upsilon)\in V_\Sigma$.

Proof (i) Suppose the statement were not true. Then we could find a sequence of monopoles γ_i such that $\mathcal{E}_I(\gamma_i)\to 0$ yet no gauge transformation maps any of the γ_i into V. In particular, since the energies of the configurations γ_i converge, then by [21, Theorem 5.1.1], a subsequence of the γ_i converges in $H^1(I\times\Sigma)$ modulo gauge. The limiting monopole must have zero energy and therefore belongs to $\mathscr{V}_I$. But this means that for some i, a gauge transformation maps γ_i into the neighborhood V, a contradiction.

(ii) We have a corresponding energy identity for arbitrary configurations (C,Υ) of $\mathfrak{C}(\Sigma)$:

$$\int_\Sigma \left(\frac{1}{4}|F_C|^2 + |\nabla_C\Upsilon|^2 + \frac{1}{4}\Big(|\Upsilon|^2+(s/2)\Big)^2 - \frac{s^2}{16}\right) = \|SW_2(C,\Upsilon)\|^2_{L^2(\Sigma)} + \|\mu(C,\Upsilon)\|^2_{L^2(\Sigma)}.$$

The proof is now the same as in (i). □

Corollary 6.2 *For every gauge invariant neighborhood V_Σ of $\mathscr{V}$ in $\mathfrak{C}^{1/2}(\Sigma)$, there exists an $\epsilon>0$ such that if γ is a monopole on $I\times\Sigma$ with $\int_I \|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)}<\epsilon$, then modulo gauge, we have $\check{\gamma}(t)\in V_\Sigma$ for all $t\in I$.*

Proof We apply the previous lemma and use that $H^1([0,1]\times\Sigma)$ embeds into the space $C^0([0,1],H^{1/2}(\Sigma))$. □

Lemma 6.3 *For every $\epsilon > 0$, there exists an $\epsilon_0 > 0$ with the following significance. Let $T \geq 1$ and let γ be a monopole such that $\int_{T-1}^{T+1} \|SW_2(\gamma(t))\|_{L^2(\Sigma)} \leq \epsilon_0$.*

(i) We have $\|SW_2(\check{\gamma}(T))\|_{L^2(\Sigma)} \leq \epsilon$.

(ii) If $\check{\gamma}(T)$ belongs to a coordinate patch $U_{\mathfrak{a}}(\delta)$ for δ sufficiently small, then $\|\check{\gamma}(T) - \mathfrak{a}\|_{H^s(\Sigma)} \leq C_s\epsilon$ for all $s \geq 1/2$.

Proof (i) Let $I = [T-1, T+1]$. By the previous lemma, if ϵ_0 is sufficiently small, then there exists a vortex $\mathfrak{a} \in \mathscr{V}$ and a gauge transformation such that $\|g^*\gamma - \gamma_{\mathfrak{a}}\|_{H^1([T-1,T+1]\times\Sigma)} \leq \epsilon'$. Since both $g^*\gamma$ and $\gamma_{\mathfrak{a}}$ are solutions to Seiberg-Witten equations on $I \times \Sigma$, one can bootstrap the regularity of $(b, \psi) := g^*\gamma - \gamma_{\mathfrak{a}}$ on the smaller cylinder $[T - 1/2, T + 1/2] \times \Sigma$ once we put (b, ψ) in Coulomb-Neumann gauge. More precisely, we find another gauge-transformation $\tilde{g}$ such that $\tilde{g}^*(g^*\gamma) - \gamma_{\mathfrak{a}} = (\tilde{b}, \tilde{\psi})$ satisfies $d^*\tilde{b} = 0$ and $*\tilde{b}|_{\partial I\times\Sigma} = 0$. Then

$$\|(\tilde{b}, \tilde{\psi})\|_{H^1(I\times\Sigma)} \leq C\|(b, \psi)\|_{H^1(I\times\Sigma)} \leq C\epsilon'$$

since the (linear) projection onto the Coulomb-Neumann slice is a bounded operator on H^1. We have

$$SW_3((\tilde{g}g)^*\gamma) - SW_3(\gamma_{\mathfrak{a}}) = \mathcal{H}_{\gamma_{\mathfrak{a}}}(\tilde{b}, \tilde{\psi}) + (\tilde{b}, \tilde{\psi})\#(\tilde{b}, \tilde{\psi}) = 0,$$

where $\mathcal{H}_{\gamma_{\mathfrak{a}}}$ is the Hessian (3.50) and $\#$ is a bilinear multiplication map. One can now bootstrap the regularity of $(\tilde{b}, \tilde{\psi})$ in the interior, since the operator $\mathcal{H}_{\gamma_{\mathfrak{a}}}$ yields interior elliptic estimates for configurations in Coulomb gauge. (We cannot obtain estimates up the boundary since we have no boundary conditions on ψ.) We obtain

$$\|(\tilde{b}, \tilde{\psi})\|_{H^s([T-1/2,T+1/2]\times\Sigma)} \leq C_s\|(\tilde{b}, \tilde{\psi})\|_{H^1(I\times\Sigma)}$$

for some constant C_s depending on $s \geq 1$. Here, we can choose C_s independent of $\mathfrak{a}$ since we can always choose $\mathfrak{a}$ from a compact subset of $\mathscr{V}$, since $\mathscr{V}$ is compact modulo gauge.

The trace theorem then gives

$$\|(\tilde{b}(T), \tilde{\psi}(T))\|_{H^{3/2}(\{T\}\times\Sigma)} \leq C\|(b, \psi)\|_{H^2([T-1/2,T+1/2]\times\Sigma)}.$$

Write $b(T) = c(T) + \beta(T)dt$. By gauge equivariance of the map SW_2 and gauge invariance of the L^2 norm, we have

$$\begin{aligned}\|SW_2(\check{\gamma}(T))\|_{L^2(\Sigma)} &= \|SW_2((\tilde{g}g)^*\check{\gamma}(T)) - SW_2(\mathfrak{a})\|_{L^2(\Sigma)}\\ &\leq \text{const} \cdot \|(c(T), \psi(T))\|_{H^1(\{T\}\times\Sigma)}\\ &\leq \text{const} \cdot \|(b, \psi)\|_{H^{3/2}([T-1/2,T+1/2]\times\Sigma)}\\ &\leq \text{const} \cdot \epsilon'.\end{aligned}$$

In the second line above, we used that $\|(c(T), \psi(T))\|_{H^1(\Sigma)}$ is small and controls $\|(c(T), \psi(T))\|_{L^4(\Sigma)}$. Choosing ϵ' small enough proves the lemma.

(ii) By (i) and Lemma 6.1(ii), it follows that $\check{\gamma}(t)$ is $H^1(\Sigma)$ close to $\mathfrak{a}$. Let $\gamma' = (\tilde{g}g)^*\gamma$ be the configuration from (i) in Coulomb-Neumann gauge relative to $\gamma_{\mathfrak{a}}$. We have that $\|\gamma' - \gamma_{\mathfrak{a}}\|_{H^s([T-1/2,T+1/2]\times\Sigma)} \leq C_s\epsilon$ for any $s \geq 1$. On the other hand, by Lemma 5.4,

$$\|\check{\gamma}(T) - \mathfrak{a}\|_{H^{s-1/2}(\{T\}\times\Sigma)} \leq \|\check{\gamma}'(T) - \mathfrak{a}\|_{H^{s-1/2}(\{T\}\times\Sigma)} \tag{6.2}$$

since $\check{\gamma}(T)$ belongs to a small neighborhood in the slice through $\mathfrak{a}$. The result now follows. □

Given a vortex $\mathfrak{a}$, below are Morse-Bott type inequalities for $CSD^{\Sigma}_{\mathfrak{a}}$ in our infinite-dimensional setting.

Lemma 6.4 *Given a smooth vortex $\mathfrak{a} \in \mathscr{V}$, there exists $\delta > 0$ such the following holds. If $(C, \Upsilon) \in U_{\mathfrak{a}}(\delta)$ then*

$$|CSD^{\Sigma}_{\mathfrak{a}}(C,\Upsilon)| \leq \text{const} \cdot \|(C,\Upsilon) - \mathfrak{a}\|^2_{H^1(\Sigma)}. \tag{6.3}$$

$$|CSD^{\Sigma}_{\mathfrak{a}}(C,\Upsilon)|^{1/2} \leq \text{const} \cdot \|\nabla_{(C,\Upsilon)} SW_2(C,\Upsilon)\|_{L^2(\Sigma)}. \tag{6.4}$$

Proof Let (C_0, Υ_0) and (C, Υ) be any two configurations and let $(c, v) = (C - C_0, \Upsilon - \Upsilon_0)$ be their difference. A simple Taylor expansion of the cubic function CSD^{Σ} shows that it satisfies

$$\begin{aligned} CSD^{\Sigma}(C_0 + c, \Upsilon_0 + v) = CSD^{\Sigma}(C_0, \Upsilon_0) + ((c,v), SW_2(C_0,\Upsilon_0)) + \\ \frac{1}{2}((c,v), \mathcal{H}_{2,(C_0,\Upsilon_0)}(c,v)) + \frac{1}{2}(v, \rho(c)v). \end{aligned} \tag{6.5}$$

Letting (C_0, Υ_0) be a vortex $\mathfrak{a}$, then since $SW_2(\mathfrak{a}) = 0$, we have from (6.5) that

$$\begin{aligned} |CSD^{\Sigma}_{\mathfrak{a}}(C,\Upsilon)| &\leq \frac{1}{2}\|((c,v), \mathcal{H}_{2,\mathfrak{a}}(c,v))\|_{L^2(\Sigma)} + \frac{1}{2}\|(c,v)\|^3_{L^3(\Sigma)} \\ &\leq \text{const}\left(\|(c,v)\|^2_{H^1(\Sigma)} + \|(c,v)\|_{H^{1/2}(\Sigma)\|}\|(c,v)\|^2_{H^1(\Sigma)}\right). \end{aligned}$$

Here, we use that $\mathcal{H}_{2,\mathfrak{a}}$ is a first order with smooth coefficients and we use the embedding $H^{1/2}(\Sigma) \hookrightarrow L^4(\Sigma) \subset L^3(\Sigma)$. The estimate (6.3) now follows from the hypotheses, which implies $\|(c,v)\|_{H^{1/2}(\Sigma)} < \delta$.

The second inequality (6.4) is a standard inequality for Morse-Bott type functions, which one can establish using an infinite-dimensional version of the Morse-Bott lemma (see [9, Chapter 4.5]). Using the same techniques of Section 4, where we analyzed local straightening maps for Banach submanifolds in various topologies, one can verify that a Morse-Bott lemma can be performed in a $H^{1/2}(\Sigma)$ neighborhood of the space of vortices. □

Remark 6.5 The standard Morse-Bott inequality (in finite dimensions) states that $|f(x)|^{1/2} \leq c|\nabla_x f|$ holds in a neighborhood of the critical set of a Morse-Bott function f. In the above, we have been a bit cavalier in our notion of the gradient, since an inner product needs to be specified. However, since the projection $\Pi_{\mathcal{K}_{\mathfrak{a}}, \mathcal{J}_{(C,\Upsilon)}} : \mathcal{K}_{(C,\Upsilon)} \to \mathcal{K}_{\mathfrak{a}}$ is an isomorphism, uniformly in the L^2 norm for $\|(C,\Upsilon) - \mathfrak{a}|_{H^{1/2}(\Sigma)}$ sufficiently small, whether we use the usual L^2 inner product or the inner product (5.30) is immaterial.

Definition 6.6 We say that the chart $U_{\mathfrak{a}}$ is a *Morse-Bott chart* for $\mathfrak{a}$ if it its closure is contained in a chart of the form $U_{\mathfrak{a}}(\delta)$, with δ sufficiently small as in Lemma 6.4.

We are interested in configurations which are in standard form with respect to a Morse-Bott coordinate chart. This is because the Morse-Bott estimates we obtain on these charts allow us to prove the usual exponential decay estimates for Morse-Bott type flows:

Lemma 6.7 *Let γ be a smooth finite energy solution to $SW_3(\gamma) = 0$ on $Y = [T, \infty) \times \Sigma$ which is in standard form with respect to a Morse-Bott chart $U_{\mathfrak{a}}$ on Y. Then we have the following:*

(i) The path $\check{\gamma}(t)$ converges in $L^2(\Sigma)$ to a vortex $\mathfrak{a}$ as $t \to \infty$ and the temporal component of γ converges in $L^2(\Sigma)$ to zero.

(ii) The energy of γ, or more precisely, the function $CSD^{\Sigma}_{\mathfrak{a}}(\check{\gamma}(t))$, decays exponentially as $t \to \infty$.

Proof Let $CSD^{\Sigma}_{\mathfrak{a}}(t) = CSD^{\Sigma}_{\mathfrak{a}}(\check{\gamma}(t))$. It is a nonnegative, nonincreasing function of t. We obtain the differential inequality

$$\frac{d}{dt}CSD^{\Sigma}_{\mathfrak{a}}(t) = -\left\langle \Pi_{\mathcal{K}_{\mathfrak{a}},\mathcal{J}_{\check{\gamma}(t)}} SW_2(\check{\gamma}(t)), \Pi_{K_{\mathfrak{a}},\mathcal{J}_{\check{\gamma}(t)}} SW_2(\check{\gamma}(t)) \right\rangle_{\mathfrak{a},\check{\gamma}(t)} \tag{6.6}$$

$$= -\|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)} \tag{6.7}$$

$$\leq -\text{const} \cdot CSD^{\Sigma}_{\mathfrak{a}}(t). \tag{6.8}$$

In the first line we used (5.31), in the second line, we used that $\Pi_{\mathcal{K}_{(C,\Upsilon)}}\Pi_{\mathcal{K}_{\mathfrak{a}},J_{\check{\gamma}(t)}} = \Pi_{\mathcal{K}_{(C,\Upsilon)}}$, and in the last line, we used (6.4). The above inequality implies that

$$CSD^{\Sigma}_{\mathfrak{a}}(t) \leq c_0 e^{-\delta_0 t} \cdot CSD^{\Sigma}_{\mathfrak{a}}(T)$$

for some c_0 and δ_0 depending on $\mathfrak{a}$. Since the space of vortices is compact modulo gauge however, we can ultimately choose c_0 and δ_0 independent of $\mathfrak{a}$.

Moreover, we have the following length estimate. First, we have

$$\begin{aligned}\|SW_2(\check{\gamma}(t))\|_{L^2(\Sigma)} &= \|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)}\|SW_2(\check{\gamma}(t))\|^{-1}_{L^2(\Sigma)} \\ &\leq c\|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)} CSD^{\Sigma}_{\mathfrak{a}}(t)^{-1/2} \\ &= -c\frac{d}{dt}CSD^{\Sigma}_{\mathfrak{a}}(t)^{1/2},\end{aligned}$$

Note Remark 6.5 in passing to the last line. The above computation makes sense for any non-stationary monopole γ, since then $SW_2(\check{\gamma}(t)) \neq 0$ for every t (otherwise, by unique continuation, we would have $SW_2(\check{\gamma}(t)) = 0$ for all t). Thus,

$$\begin{aligned}\|\check{\gamma}(T_0) - \check{\gamma}(T_1)\|_{L^2(\Sigma)} &\leq \int_{T_0}^{T_1} \left\|\frac{d}{dt}\check{\gamma}(t)\right\|_{L^2(\Sigma)} dt \\ &\leq c\int_{T_0}^{T_1} \|SW_2(\check{\gamma}(t))\|_{L^2(\Sigma)} dt \\ &\leq c'\left(CSD^{\Sigma}_{\mathfrak{a}}(T_0)^{1/2} - CSD^{\Sigma}_{\mathfrak{a}}(T_1)^{1/2}\right).\end{aligned} \tag{6.9}$$

Since $CSD^{\Sigma}_{a}(t)$ is decreasing to zero, then the $\check{\gamma}(t)$ form a Cauchy sequence in $L^2(\Sigma)$. In particular, the path $\check{\gamma}(t)$ converges to a limit, which must be a vortex. □

We need two more important facts. First, we want to show that we can satisfy the hypothesis of the previous lemma, namely that given a monopole with small enough energy, one can always find a coordinate patch about a vortex and a gauge transformation that places the monopole into standard form for all future time with respect to the coordinate patch. Secondly, we want to show that not only does a monopole in standard form yield a path of configurations convergent in $L^2(\Sigma)$ to a vortex but that the monopole itself on Y converges exponentially in all H^k Sobolev norms on Y. This is guaranteed by the following lemma, whose proof we mostly relegate to [28] because of its technical nature:

Lemma 6.8 *There exists an $\epsilon_0 > 0$ with the following significance.*

(i) If γ is a monopole such that $\int_T^\infty \|SW_2(\check{\gamma}(t))\|^2_{L^2(\Sigma)}dt = \epsilon < \epsilon_0$, then there exists a Morse-Bott coordinate patch $U_{\mathfrak{a}}$ and a gauge transformation g such that $\gamma' = g^*\gamma$ is in standard form with respect to $U_{\mathfrak{a}}$ on $[T,\infty)\times\Sigma$.

(ii) There exists a $\delta_0 > 0$ such if $0 < \delta < \delta_0$, then

$$\|\gamma'\|_{H^s([T+1,\infty)\times\Sigma)} \leq C_s C_\epsilon e^{-\delta t} \tag{6.10}$$

for every $s \geq 0$. Here C_s is a constant depending on s and C_ϵ is a constant that can be taken arbitrarily small for ϵ sufficiently small.

Proof (i) By Lemma 6.1, for ϵ_0 sufficiently small, we can find a gauge transformation g on $[T,T+1]\times\Sigma$ such that $g^*\gamma$ is in standard form with respect to some Morse-Bott patch $U_{\mathfrak{a}}$. The key step is to show that g can be extended to all of $[T,\infty)\times\Sigma$ in such a way that the resulting gauge transformation places γ in standard form for all future time. However, Lemma 6.3 together with the same arguments as in [28, Theorem 4.3.1] shows that this is the case for ϵ_0 sufficiently small.

(ii) By (i) and Lemma 6.3, we know that $\sup_{t\geq T+1}\|\check{\gamma}'(t)\|_{H^k(\Sigma)} \leq C_k C_\epsilon$. Now standard exponential decay arguments, e.g. [28, Lemma 5.4.1][5] yields the desired conclusion for $s=0$. For $s>0$, we use the fact that one bootstrap elliptic estimates in the standard form gauge so that L^2 exponential decay gives us H^s decay on the cylinder. The arguments are formally similar to those of [28, Lemma 3.3.2]. We omit the details. Note that we can take δ_0 independent of $\mathfrak{a}$ since the vortex moduli space $\mathcal{V}$ is compact. □

7 The Finite Energy Moduli Space on $[0,\infty)\times\Sigma$

In this section, we use the results developed in the previous section to prove our main results concerning the space of finite energy monopoles on $[0,\infty)\times\Sigma$. From Lemma 6.8, we see that modulo gauge, any finite energy monopole converges exponentially to a vortex. This result depends crucially on the Morse-Bott framework[6] of the previous section and it yields for us the following two bits of information. First, it tells us that the right choice of function

[5] Note this lemma is a more general statement than we need, since in our Morse-Bott situation, the center manifold is simply the critical manifold.

[6] For comparison, in [28], one does not always get exponential decay in the instanton case due to Morse-Bott degenerate critical points.

spaces to consider on the cylinder are the exponentially weighted Sobolev spaces. Second, it suggests that the topology of our monopole spaces is related to the topology of the vortex moduli spaces at infinity. Our main theorems of this section, Theorems 7.2, 7.6, and 7.7, confirm this latter expectation.

We begin with the appropriate function space setup. Although we will ultimately only need to work with smooth monopoles in the case when Y is a manifold with cylindrical ends (and without boundary), our preliminary analysis here on the ends requires us to work in the general Hilbert space setting. On the noncompact space $Y=[0,\infty)\times\Sigma$, we must a priori work with local Sobolev spaces $H^s_{\mathrm{loc}}(Y)$, that is, the topological vector space of functions on Y that belong to $H^s(K)$ for every compact domain $K\subset Y$. We let $\mathfrak{C}^s_{\mathrm{loc}}(Y)$ denote the $H^s_{\mathrm{loc}}(Y)$ completion of the smooth configuration space on Y. Then the space of all finite energy monopoles in $\mathfrak{C}^s_{\mathrm{loc}}(Y)$ is given by

$$\mathfrak{M}^s=\mathfrak{M}^s(Y)=\{\gamma\in\mathfrak{C}^s_{\mathrm{loc}}(Y):\mathcal{E}(\gamma)<\infty\}.$$

We topologize this space in the $H^s_{\mathrm{loc}}(Y)$ topology and also by requiring that the energy be a continuous function. Likewise, for any $E>0$, we can define the space

$$\mathfrak{M}^s_E=\{\gamma\in\mathfrak{C}^s_{\mathrm{loc}}(Y):\mathcal{E}(\gamma)<E\}.$$

of $H^s_{\mathrm{loc}}(Y)$ monopoles that have energy less than E.

These spaces, being merely the spaces which a priori contains all the monopoles of interest, are much too large to be of use. Of course, as we have mentioned, we can always find a gauge in which a finite energy configuration decays exponentially in every Sobolev norm at infinity. So for any $\delta\in\mathbb{R}$ and nonnegative integer $s\geq 0$, define $H^{s;\delta}(Y)$ to be the closure of $C^\infty_0(Y)$ in the norm

$$\|f\|_{H^{s;\delta}(Y)}=\|e^{\delta t}f\|_{H^s(Y)}.$$

Thus, for $\delta>0$, the weight $e^{\delta t}$ forces exponential decay of our functions; for $\delta<0$, we allow exponential growth. Using this topology, we can topologize the space $\mathcal{T}=\Omega^1(Y;i\mathbb{R})\oplus\Gamma(\mathcal{S})$ (the tangent space to the smooth configuration space on Y when Y was compact) in the $H^{s;\delta}(Y)$ topology to obtain $\mathcal{T}^{s;\delta}$. For $\delta>0$, we can then define the corresponding space

$$\mathfrak{C}^{s;\delta}(Y)=\{\gamma:\gamma-\gamma_{\mathfrak{a}}\in\mathcal{T}^{s;\delta}\text{ for some }\mathfrak{a}\in\mathscr{V}^s\}$$

of configurations that decay exponentially to some $H^s(\Sigma)$ vortex $\mathfrak{a}\in\mathscr{V}^s:=H^s\mathscr{V}$. In particular, if $s\geq 2$, all configurations in $\mathfrak{C}^{s;\delta}(Y)$ are pointwise bounded. From now on, we will assume s is an integer and $s\geq 2$ unless otherwise stated. We give $\mathfrak{C}^{s;\delta}(Y)$ the topology of $\mathcal{T}^{s;\delta}\times\mathscr{V}^s$ in the obvious way. In particular, observe that $\mathfrak{C}^{s;\delta}(Y)$ is a Hilbert manifold. Define the map

$$\begin{aligned}\tilde{\partial}_\infty:\mathfrak{C}^{s;\delta}(Y)&\to\mathscr{V}^s\\ \gamma&\mapsto\lim_{t\to\infty}\check{\gamma}(t).\end{aligned}\tag{7.1}$$

Then the tangent space to $\mathfrak{C}^{s;\delta}(Y)$ at γ is the space

$$T_\gamma \mathfrak{C}^{s;\delta}(Y) = \mathcal{T}^{s;\delta} \oplus T_{\tilde{\partial}_\infty(\gamma)} \mathscr{V}^s.$$

Given $\gamma \in \mathfrak{M}^s$, the gauge transformation which sends γ to an element of $\mathfrak{C}^{s;\delta}(Y)$, being only required to satisfy a condition at infinity, can be taken to be identically one near $\Sigma = \partial Y$. It follows that to study the space $\mathfrak{M}^s$ and its boundary values on $\mathfrak{C}^{s-1/2}(\Sigma)$, it suffices to study the space

$$\mathfrak{M}^{s;\delta} = \mathfrak{M}^s \cap \mathfrak{C}^{s;\delta}(Y),$$

for $\delta > 0$ small.

We now consider the above setup modulo all gauge transformations. When $\delta > 0$, the exponential decay of configurations allows multiplication to be possible and we can define an exponentially weighted gauge group accordingly. Namely, we define $\mathcal{G}^{s+1;\delta}(Y)$ to be the Hilbert Lie group of gauge transformations such that $g - 1$ belongs to $H^{s+1;\delta}(Y)$. This group acts smoothly on $\mathfrak{C}^{s;\delta}(Y)$ and we can form the quotient space

$$\mathfrak{B}^{s;\delta}(Y) = \mathfrak{C}^{s;\delta}(Y)/\mathcal{G}^{s+1;\delta}(Y).$$

It is a smooth Hilbert manifold away from the reducible configurations, which we can ignore when studying the monopole moduli space due to the nontriviality of the spin^c structure $\mathfrak{s}$ on Y. Let

$$M^s = M^s(Y) = \mathfrak{M}^{s;\delta}(Y)/\mathcal{G}^{s+1;\delta}(Y) \subset \mathfrak{B}^{s;\delta}(Y) \tag{7.2}$$

denote the moduli space of gauge equivalence classes of exponentially decaying monopoles in $\mathfrak{M}^{s;\delta}$. By our exponential decay results, M^s, topologized as a subspace of $\mathfrak{B}^{s;\delta}(Y)$, is also (topologically) the quotient space of $\mathfrak{M}^s$ by the group of $H^{s+1}_{\mathrm{loc}}(Y)$ gauge transformations on Y. (Here, it is key that $\mathfrak{M}^s$ is topologized with the energy functional.) Observe that the definition of M^s is independent of δ for $\delta > 0$ sufficiently small as a consequence of Lemma 6.8.

Remark 7.1 In Part I, we considered only partially gauge-fixed monopole spaces. In our scenario, the analogous space would be the space

$$\mathcal{M}^{s;\delta} = \{\gamma \in \mathfrak{M}^{s;\delta} : d^*(\gamma - \gamma_{\mathrm{ref}}) = 0\} \tag{7.3}$$

for some smooth reference configuration γ_{ref} pulled back from a configuration on $\mathfrak{C}(\Sigma)$. A global Coulomb gauge requires only using gauge transformations that are the identity on the boundary, so that the space of boundary values of Coulomb gauge-fixed monopoles live on the configuration space $\mathfrak{C}^{s-1/2}(\Sigma)$ on the boundary and not the quotient configuration space. Thus, in Part I we work in Coulomb gauge because of its importance for the analysis of Part III, where we want gauge-invariant Lagrangian submanifolds of the boundary configuration space to yield for us boundary conditions for the Seiberg-Witten equations. In our present setting, the gauge-freedom on the boundary Σ is of no interest to us, and so we work in the usual setting of quotienting by all gauge transformations. Of course, there is no loss of information in deciding whether to work with the configuration space on Σ or its quotient by gauge transformations.

To work modulo gauge, we want to obtain tangent space decompositions arising from the infinitesimal gauge action, as in Section 3.2, but on weighted spaces. Thus, for $(B,\Psi)\in\mathfrak{C}^{s;\delta'}(Y)$ with $\delta'>0$, we can define the operators

$$\begin{aligned}\mathbf{d}_{(B,\Psi)}: H^{s+1;\delta}(Y;i\mathbb{R}) &\to \mathcal{T}^{s;\delta}\\ \xi &\mapsto (-d\xi,\xi\Psi)\\ \mathbf{d}^*_{(B,\Psi)}:\mathcal{T}^{s;\delta}&\to\mathcal{T}^{s-1;\delta}\\ (b,\psi)&\mapsto -d^*b+i\mathrm{Re}\,(i\Psi,\psi).\end{aligned}$$

for $\delta\le\delta'$. We then obtain the subspaces

$$\begin{aligned}\mathcal{J}^{s;\delta}_{(B,\Psi)}&=\mathrm{im}\,\mathbf{d}_{(B,\Psi)}\\ \mathcal{K}^{s;\delta}_{(B,\Psi)}&=\ker\mathbf{d}^*_{(B,\Psi)}\end{aligned}$$

of $\mathcal{T}^{s;\delta}_{(B,\Psi)}$. Likewise, we can define the spaces

$$\mathcal{J}^{s;\delta}_{(B,\Psi),t}=\{(-d\xi,\xi\Psi):\xi\in H^{s+1;\delta}(Y;i\mathbb{R}),\ \xi|_\Sigma=0\},\tag{7.4}$$

$$\mathcal{K}^{s;\delta}_{(B,\Psi),n}=\{(b,\psi)\in\mathcal{K}^{s;\delta}_{(B,\Psi)}:*b|_\Sigma=0\},\tag{7.5}$$

$$\mathcal{C}^{s;\delta}=\{(b,\psi)\in\mathcal{T}^{s;\delta}:d^*b=0\},\tag{7.6}$$

where $\Sigma=\{0\}\times\Sigma$ is the boundary of Y. By standard Fredholm theory on weighted spaces (see [25]), we can obtain a weighted decomposition

$$\mathcal{T}^{s;\delta}=\mathcal{J}^{s;\delta}_{(B,\Psi)}\oplus\mathcal{K}^{s;\delta}_{(B,\Psi),n}\tag{7.7}$$

for (B,Ψ) irreducible, proceeding mutatis mutandis as in Lemma 3.4 (with $\delta\neq0$ sufficiently small). This is summarized in Lemma 7.3.

The operator (7.1) induces the following smooth map on the quotient space,

$$\begin{aligned}\partial_\infty:\mathfrak{B}^{s;\delta}(Y)&\to\mathcal{V}\\ [\gamma]&\mapsto\lim_{t\to\infty}[\check{\gamma}(t)]\end{aligned}\tag{7.8}$$

Given any irreducible $\gamma\in\mathfrak{C}^s(Y)$, from (7.7), we have that the tangent space to $[\gamma]$ of $\mathfrak{B}^{s;\delta}(Y)$ can be identified with

$$T_{[\gamma]}\mathfrak{B}^{s;\delta}(Y)\cong\mathcal{K}^{s;\delta}_{\gamma,n}\cap T_{\partial_\infty[\gamma]}\mathcal{V}.\tag{7.9}$$

The map (7.8) restricts to a map

$$\partial_\infty:M^s(Y)\to\mathcal{V}$$

mapping a monopole to its asymptotic vortex on Σ. Given a vortex $[\mathfrak{a}]\in\mathcal{V}$, we can define

$$M^s_{[\mathfrak{a}]}=\{[\gamma]\in M^s(Y):\partial_\infty[\gamma]=[\mathfrak{a}]\}\tag{7.10}$$

the moduli space of monopoles that converge to $[\mathfrak{a}]$.

We are now in the position to state our main result. We have the (tangential) restriction map

$$\begin{aligned} r_\Sigma : \mathfrak{C}^{s;\delta}(Y) &\to \mathfrak{C}^{s-1/2}(\Sigma) \\ (B,\Psi) &\mapsto (B,\Psi)|_\Sigma. \end{aligned}$$

Letting

$$\mathfrak{B}^s(\Sigma) = \mathfrak{C}^s(\Sigma)/\mathcal{G}^{s+1}(\Sigma) \tag{7.11}$$

denote the quotient configuration space on Σ, the restriction map r_Σ descends to the quotient space:

$$r_\Sigma : \mathfrak{B}^{s;\delta}(Y) \to \mathfrak{B}^{s-1/2}(\Sigma). \tag{7.12}$$

Let

$$\mathfrak{B}^s_\mu(\Sigma) = \mu^{-1}(0)/\mathcal{G}^{s+1}(\Sigma) \subset \mathfrak{B}^s(\Sigma) \tag{7.13}$$

denote the symplectically reduced space associated to the moment map μ. We have the following theorem, which geometrically, is the statement that $M^s(Y)$ is the (infinite-dimensional) stable manifold to the space of vortices $\mathcal{V}$ under the Seiberg-Witten flow.

Theorem 7.2 *(Finite Energy Moduli Space) Fix a* spin^c *structure* $\mathfrak{s}$ *on* Σ *and let* $s \geq 2$ *be an integer. Let* $d = \langle c_1(\mathfrak{s}), \Sigma\rangle$ *be nonzero. Then the following holds:*

(i) The moduli space $M^s(Y)$ *is naturally a smooth Hilbert manifold*[7] *of* $\mathfrak{B}^{s;\delta}(Y)$, *for* $\delta > 0$ *sufficiently small.*

(ii) The map $r_\Sigma : M^s(Y) \to \mathfrak{B}^{s-1/2}(\Sigma)$ *is a diffeomorphism onto its image, which is a coisotropic submanifold of the symplectically reduced space* $\mathfrak{B}^{s-1/2}_\mu(\Sigma)$. *Given any* $[\gamma] \in M^s(Y)$, *the annihilator of the coisotropic space* $r_\Sigma(T_{[\gamma]}M^s(Y))$ *is the space* $r_\Sigma\big(T_{[\gamma]}M^s_{\partial_\infty[\gamma]}(Y)\big)$.

(iii) Both $M^s(Y)$ *and* $r_\Sigma(M^s(Y))$ *are complete.*

In regarding $M^s = M^s(Y)$ as the stable manifold to the space of vortices at infinity, we see that it is the union of the $M^s_{[\mathfrak{a}]}$, each of which is the stable manifold to $[\mathfrak{a}] \in \mathcal{V}$, as $[\mathfrak{a}]$ varies over the symplectic set of critical points $\mathcal{V}$. This geometric picture clarifies the symplectic nature of (ii) in the above.

Because of the infinite-dimensional nature of the objects involved, the proof of the above theorem requires some care. We first prove a few lemmas. The first lemma below is an adaptation of the relevant results of Part I on compact 3-manifolds adapted to the cylindrical case. Here, the adaptation arises from considering weighted spaces, with a small non-zero weight parameter δ.

Lemma 7.3 *Let* $s \geq 2$ *and* $\gamma \in \mathfrak{C}^{s;\delta'}(Y)$ *where* $\delta' > 0$. *Then for* $\delta > 0$ *sufficiently small, the following hold:*

[7] From now on, we will always regard $M^s(Y)$ as endowed with this topology. As mentioned, it is homeomorphic to the quotient of $\mathfrak{M}^s$ by $G^{s+1}_{\mathrm{loc}}(Y)$, but the latter does not come with an a priori smooth manifold structure.

(i) We have the following decompositions for $1 \le s' \le s$*:*

$$\mathcal{T}^{s';\pm\delta} = \mathcal{J}^{s';\pm\delta}_{\gamma,t} \oplus \mathcal{C}^{s';\pm\delta} \tag{7.14}$$

$$\mathcal{T}^{s';\pm\delta} = \mathcal{J}^{s';\pm\delta}_{\gamma,t} \oplus \mathcal{K}^{s';\pm\delta}_{\gamma}. \tag{7.15}$$

If γ *is not reducible, then we also have*

$$\mathcal{T}^{s';\pm\delta} = \mathcal{J}^{s';\pm\delta}_{\gamma} \oplus \mathcal{K}^{s';\pm\delta}_{\gamma,n}. \tag{7.16}$$

(ii) Let $SW_3(\gamma) = 0$*. Then the Hessian operator* $\mathcal{H}_\gamma : \mathcal{T}^{s;\pm\delta} \to \mathcal{K}^{s-1;\pm\delta}$ *is surjective.*

Proof Using the Fredholm theory for elliptic operators on weighted spaces of [25], the proof of this lemma proceeds mutatis mutandis as in Lemmas 3.4, 3.16, and 4.1, since the elliptic methods there adapt to weighted spaces for weights on the complement of a discrete set. □

In Part I, we made ample use of the augmented Hessian operator $\widetilde{\mathcal{H}}_\gamma$, defined via (3.54), an elliptic formally self-adjoint operator that extends the Hessian operator $\mathcal{H}_\gamma$ and which is naturally tied to (global) Coulomb gauge-fixing. On the other hand, had we chosen to gauge fix into the subspace $\mathcal{K}_\gamma$ instead of the Coulomb slice $\mathcal{C}$, we could have defined a different elliptic extension (which is more natural in some sense, since $\mathcal{K}_\gamma$ is orthogonal to the infinitesimal action of the gauge group identically one on the boundary). Thus, we can define, using the same terminology of [21], the *extended Hessian*

$$\overline{\mathcal{H}}_\gamma := \begin{pmatrix} \mathcal{H}_\gamma & \mathbf{d}_\gamma \\ \mathbf{d}^*_\gamma & 0 \end{pmatrix} : \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R}) \to \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R}), \tag{7.17}$$

Thus we use the operators $\mathbf{d}_\gamma$ and $\mathbf{d}^*_\gamma$ in $\overline{\mathcal{H}}_\gamma$ instead of the Coulomb slice operators $-d$ and $-d^*$ in $\widetilde{\mathcal{H}}_\gamma$. Since $\overline{\mathcal{H}}_\gamma$ and $\widetilde{\mathcal{H}}_\gamma$ differ by a zeroth order term, then on a compact manifold, such a lower order term is a compact perturbation. Hence in Part I, it is immaterial whether we work with the extended or augmented Hessian, and we ultimately chose to work with the latter, since Coulomb gauge fixing can be done globally on the configuration space and is therefore convenient. However, in our non-compact cylindrical situation, bounded operators are no longer compact perturbations of elliptic operators, and so now our choice of elliptic extension of $\mathcal{H}_\gamma$ becomes important. As one might naturally expect, $\overline{\mathcal{H}}_\gamma$ is the proper operator to consider. The significance of this choice is reflected in Lemma 7.5.

Define the weighted augmented spaces

$$\widetilde{\mathcal{T}}^{s;\delta} = \mathcal{T}^{s;\delta} \oplus H^{s;\delta}(Y; i\mathbb{R})$$

for $\delta \in \mathbb{R}$. For $\gamma \in \mathfrak{C}^{\delta'}(Y)$ and $\delta \le \delta'$, we thus get a first order formally self-adjoint elliptic operator

$$\overline{\mathcal{H}}_\gamma : \widetilde{\mathcal{T}}^{s;\delta} \to \widetilde{\mathcal{T}}^{s-1;\delta}.$$

In Part I, a method known as the "invertible double", which we alluded to in Proposition 15.18, is a fundamental tool in showing that the space of boundary values of kernel of $\widetilde{\mathcal{H}}_\gamma$ (and hence also of $\overline{\mathcal{H}}_\gamma$) on a compact 3-manifold yields a Lagrangian subspace of the boundary data space. Here, the same methods can be used, only now we have a slightly different situation due to the weights. Nevertheless, this invertible double technique is what allows us to obtain symplectic information for the boundary data of the kernel of augmented Hessian in the cylindrical case.

Recall that $\widetilde{\mathcal{T}}_\Sigma = \mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R})$ is the full boundary value space of $\widetilde{\mathcal{T}}$, with the full restriction map $r : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}_\Sigma$ given by (3.57). Extending to Sobolev spaces, we have $r : \widetilde{\mathcal{T}}^{s;\delta} \to \widetilde{\mathcal{T}}_\Sigma^{s-1/2}$ for $s > 1/2$. We also have the complex structure $\tilde{J} = \tilde{J}_\Sigma : \widetilde{\mathcal{T}}_\Sigma \to \widetilde{\mathcal{T}}_\Sigma$, given by (3.79), which extends the complex structure J on $\mathcal{T}_\Sigma$ and which is compatible with the product symplectic form (3.78) on $\widetilde{\mathcal{T}}_\Sigma$. As in Part I, symplectic data on $\mathcal{T}_\Sigma$ is obtained from symplectic data on $\widetilde{\mathcal{T}}_\Sigma$ via symplectic reduction with respect to the coisotropic space $\mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus 0$. Hence, we first study symplectic data on $\widetilde{\mathcal{T}}_\Sigma$, where we can use elliptic methods, in particular, the invertible double method.

Lemma 7.4 *(Weighted Invertible Double) Let $s \geq 2$ and let $\delta \neq 0$ be sufficiently small. Let $\gamma \in \mathcal{M}^{s;\delta}$. Define*

$$\widetilde{\mathcal{T}}^{s;\pm\delta} \oplus_{\tilde{J}} \widetilde{\mathcal{T}}^{s;\pm\delta} = \{(x,y) \in \widetilde{\mathcal{T}}^{s;\pm\delta} \oplus \widetilde{\mathcal{T}}^{s;\pm\delta} : r(x) = \tilde{J}r(y)\},$$

Then we have the following:

(i) The "doubled operator"

$$\overline{\mathcal{H}}_\gamma \oplus \overline{\mathcal{H}}_\gamma : \widetilde{\mathcal{T}}^{s;\delta} \oplus_{\tilde{J}} \widetilde{\mathcal{T}}^{s;-\delta} \to \widetilde{\mathcal{T}}^{s-1;\delta} \oplus \widetilde{\mathcal{T}}^{s-1;-\delta}$$

is an isomorphism.

(ii) The space $r(\ker \overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;\delta}})$ is an isotropic subspace of $\widetilde{\mathcal{T}}_\Sigma^{s-1/2}$. Its symplectic annihilator is the coisotropic subspace $r(\ker \overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;-\delta}})$.

Proof (i) One can easily construct a parametrix for the double using the methods of [3]. This shows that the double is Fredholm. Here $\delta \neq 0$ small is needed because of our Morse-Bott situation at infinity. To see that the double is injective, if $u = (u_+, u_-) \in \widetilde{\mathcal{T}}^{s;\delta} \oplus_{\tilde{J}} \widetilde{\mathcal{T}}^{s;-\delta}$ belongs to the kernel of the double, then

$$0 = (u_+, \overline{\mathcal{H}}_\gamma u_-)_{L^2(Y)} - (\overline{\mathcal{H}}_\gamma u_+, u_-)_{L^2(Y)} = -\int_\Sigma \Big(r(u_+), \tilde{J}r(u_-)\Big).$$

The second equality is Green's formula for $\overline{\mathcal{H}}_\gamma$, where $\Sigma = \partial Y$. This formula is justified since u_+ decays exponentially while u^- is at most bounded since δ is small (see (7.28)), so that there is no contribution from infinity. On the other hand, since $r(u_+) = \tilde{J}r(u_-)$, we conclude that

$$\int_\Sigma |u^+|^2 = \int_\Sigma |u^-|^2 = 0.$$

Thus $u = 0$ and so the double is injective. Integration by parts and the same argument shows that the orthogonal complement of the range of the double is zero. Thus, the double is invertible.

(ii) Green's formula above shows that $r(\ker \overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;\delta}})$ is isotropic and that it annihilates $r(\ker \overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;-\delta}})$. It remains to show that the annihilator of $r(\ker \overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;\delta}})$ is precisely

$r(\ker\overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;-\delta}})$, for which it suffices to show that $r(\ker\overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;\delta}})$ and $\tilde{J}r(\ker\overline{\mathcal{H}}_\gamma|_{\widetilde{\mathcal{T}}^{s;-\delta}})$ are (orthogonal) complements. This however follows from (i) and the same method of proof of [4, Proposition 5.12]. □

The above lemma remains true with $\overline{\mathcal{H}}_\gamma$ replaced with $\widetilde{\mathcal{H}}_\gamma$. We need to use $\overline{\mathcal{H}}_\gamma$ for the next lemma however:

Lemma 7.5 *Let $\gamma_{\mathfrak{a}}$ be a translation-invariant vortex.*

(i) We can write

$$\overline{\mathcal{H}}_{\gamma_{\mathfrak{a}}} = \tilde{J}\left(\frac{d}{dt} + \overline{\mathsf{B}}_{\mathfrak{a}}\right) \tag{7.18}$$

as in (3.62), where $\overline{\mathsf{B}}_{\mathfrak{a}} : \widetilde{\mathcal{T}}_\Sigma \to \widetilde{\mathcal{T}}_\Sigma$ is a time-independent first self-adjoint operator.

(ii) We have $\tilde{J}\overline{\mathsf{B}}_{\mathfrak{a}} = -\overline{\mathsf{B}}_{\mathfrak{a}}\tilde{J}$.

(iii) We have

$$\ker\overline{\mathsf{B}}_{\mathfrak{a}} = \{(c,\upsilon,0,0)\in\widetilde{\mathcal{T}}_\Sigma : (c,\upsilon)\in T_{\mathfrak{a}}\mathscr{V},\ \mathbf{d}^*_{\mathfrak{a}}(c,\upsilon)=0\} \tag{7.19}$$

is isomorphic to the tangent space to the vortex moduli space $\mathcal{V}$ at $[\mathfrak{a}]$.

Proof (i-ii) By the analysis of Section 3.3, we know that letting $\gamma_{\mathfrak{a}} = (B,\Psi)$, the Dirac operator $\overline{\mathcal{H}}_{\gamma_{\mathfrak{a}}}$ is the sum of $D_{\mathrm{dgc}}\oplus D_B$ and some zeroth order terms. With respect to the Clifford multiplication defining the Dirac operator $D_{\mathrm{dgc}}\oplus D_B$, the operator $\tilde{J}$ is Clifford multiplication by the inward normal $-\partial_t$. It follows that $\tilde{J}$ anticommutes with the tangential boundary operator of the cylindrical Dirac operator $D_{\mathrm{dgc}}\oplus D_B$. Thus, to prove (i) and (ii), we need only check that $\tilde{J}$ anti-commutes with the zeroth order symmetric operator $T_{\mathfrak{a}} := \overline{\mathcal{H}}_{\gamma_{\mathfrak{a}}} - D_{\mathrm{dgc}}\oplus D_B$.

Let us temporarily write a general element of $\widetilde{\mathcal{T}}_\Sigma$ as $(c,\beta,\alpha,(\upsilon_+,\upsilon_-))$, where $(\upsilon_+,\upsilon_-)\in\Gamma(\mathcal{S}_\Sigma)$, and where $(c,\beta,\alpha)\in\Omega^1(\Sigma;i\mathbb{R})\oplus\Omega^0(\Sigma;i\mathbb{R})\oplus\Omega^0(\Sigma;i\mathbb{R})$ corresponding to the decomposition of $\big(\Omega^1(Y;i\mathbb{R})\oplus\Omega^0(Y;i\mathbb{R})\big)|_\Sigma$ given by Lemma 3.8. (Thus, β is the normal component of a 1-form on Y and α corresponds to $\Omega^0(Y;i\mathbb{R})|_\Sigma$). Writing the time-independent spinor Ψ as $\Psi = \Upsilon = (\Upsilon_+,\Upsilon_-)$ with respect to (5.2), define

$$\begin{aligned} T_\Upsilon : \Gamma(\mathcal{S}_\Sigma) &\to \Omega^1(\Sigma;i\mathbb{R}) && (7.20)\\ (\upsilon_+,\upsilon_-) &\mapsto (\bar{\upsilon}_+\Upsilon_- + \bar{\Upsilon}_+\upsilon_-) + (\upsilon_+\bar{\Upsilon}_- + \Upsilon_+\bar{\upsilon}_-), && (7.21)\end{aligned}$$

which is obtained from equations (5.4) and (5.5) by linearizing the spinor terms. Then one can check that the matrix for $T_{\mathfrak{a}}$ is given by

$$T_{\mathfrak{a}} = \begin{pmatrix} 0 & 0 & 0 & T_\Upsilon(\cdot) \\ 0 & 0 & 0 & i\mathrm{Re}\,(i\Upsilon,\cdot) \\ 0 & 0 & 0 & i\mathrm{Re}\,(i\rho(-dt)\Upsilon,\cdot) \\ \rho(\cdot)\Upsilon & \rho\big(\cdot(-dt)\big)\Upsilon & (\cdot)\Upsilon & 0 \end{pmatrix}.$$

Here, we have identified $\widetilde{\mathcal{T}}$ with $\Gamma([0,\infty),\widetilde{\mathcal{T}}_\Sigma)$. The term $i\mathrm{Re}\,(i\rho(-dt)\Upsilon,\cdot)$ appearing in $T_{\mathfrak{a}}$ is precisely the term arising from linearizing the spinor terms in the moment map equation (5.3). A simple computation shows that $\tilde{J}T_{\mathfrak{a}} = -T_{\mathfrak{a}}\tilde{J}$.

(iii) From (i), elements of $\ker \overline{\mathsf{B}}_{\mathfrak{a}}$ are precisely those elements of $\ker \overline{\mathcal{H}}_{\gamma_{\mathfrak{a}}}$ that are time-independent. By Lemma 5.3(v), the linearization of the equation $SW_3(\gamma) = 0$, in temporal gauge at a vortex $\gamma = \gamma_{\mathfrak{a}}$, yields the linearization of the vortex equations at $\mathfrak{a}$. The gauge-fixing operator $\mathbf{d}^*_{\gamma_{\mathfrak{a}}}$ on Y for time-independent configurations becomes the gauge-fixing operator $\mathbf{d}^*_{\mathfrak{a}}$ on Σ. It is now clear that any element of $\ker \overline{\mathsf{B}}_{\mathfrak{a}}$ with vanishing β and α components, i.e. which belongs to $\mathcal{T}_\Sigma \subset \widetilde{\mathcal{T}}_\Sigma$, is precisely the right-hand side of (7.19).

Conversely, suppose $(c, \beta, \alpha, \upsilon) \in \ker \overline{\mathsf{B}}_{\mathfrak{a}}$. Linearizing (5.4)–(5.6), dropping time derivatives, and adding in the $\mathbf{d}_{\mathfrak{a}}\alpha$ term, we find that

$$\mathcal{H}_{2,\mathfrak{a}}(c, \upsilon) - J\mathbf{d}_{\mathfrak{a}}\beta + \mathbf{d}_{\mathfrak{a}}\alpha = 0.$$

Recall that $J = (-\check{*}, \rho(\partial_t))$ is the compatible complex structure for the symplectic form ω on $\mathcal{T}_\Sigma$. All three terms in the above however are orthogonal to each other, since the tangent space to the gauge group is isotropic and since Proposition 5.1(iv) holds. It follows that $J\mathbf{d}_{\mathfrak{a}}\beta = \mathbf{d}_{\mathfrak{a}}\alpha = 0$, whence $\beta = \alpha = 0$ since $\Psi \neq 0$. We now have the equality (7.19). □

Proof of Theorem 7.2: (i) We prove that $\mathcal{M}^{s;\delta}$ is a Hilbert manifold by showing that it is the zero set of a section of a Hilbert bundle that is transverse to zero. We have the exponentially decaying space $\mathcal{K}^{s-1;\delta}_\gamma$ for every configuration γ on Y. Just as in Proposition 3.5, since $\mathcal{K}^{s-1;\delta}_\gamma$ varies continuously with γ, we may form the bundle $\mathcal{K}^{s-1;\delta}(Y) \to \mathfrak{C}^{s;\delta}(Y)$ whose fiber over every $\gamma \in \mathfrak{C}^{s;\delta}(Y)$ is the Hilbert space $\mathcal{K}^{s-1;\delta}(Y)$.

We can interpret SW_3 as a section of the bundle $\mathcal{K}^{s-1;\delta}(Y)$, i.e.,

$$SW_3 : \mathfrak{C}^{s;\delta}(Y) \to \mathcal{K}^{s-1;\delta}(Y) \tag{7.22}$$

Note that the range of SW_3 really is contained in the exponentially decaying space $\mathcal{K}^{s;\delta}(Y)$. Indeed, for any constant vortex $\gamma_{\mathfrak{a}}$ induced from $\mathfrak{a} \in \mathscr{V}^s$ and any $x \in \mathcal{T}^{s;\delta}$, we have

$$SW_3(\gamma_{\mathfrak{a}} + x) = \mathcal{H}_{\gamma_{\mathfrak{a}}}x + x\sharp x.$$

Since $\delta > 0$ and $s \geq 2$, multiplication is bounded on $\mathcal{T}^{s;\delta}$ and so in particular, $x\sharp x \in \mathcal{T}^{s-1;\delta}$. Lemma 7.3(ii) implies (7.22) is transverse to the zero section, whence $\mathfrak{M}^{s;\delta} = SW_3^{-1}(0)$ is a smooth Hilbert submanifold of $\mathfrak{C}^{s;\delta}(Y)$. Since there are no reducibles, $\mathcal{G}^{s+1;\delta}(Y)$ acts freely, and so $M^s \cong \mathfrak{M}^{s;\delta}/\mathcal{G}^{s+1;\delta}(Y)$ has the structure of a smooth Hilbert submanifold of $\mathfrak{B}^{s;\delta}(Y)$.

(iii) The space M^s is obviously complete since it is the quotient of $\mathfrak{M}^{s;\delta}$, which is complete as it is the zero set of a continuous map. To show that $r_\Sigma(M^s)$ is complete, we have to show that any sequence in $r_\Sigma(M^s)$ which forms a Cauchy sequence in $\mathfrak{B}^{s-1/2}(\Sigma)$ converges to an element of $r_\Sigma(M^s)$. Since $s - 1/2 \geq 1/2$, if a sequence converges in $H^{s-1/2}(\Sigma)$, its values under CSD^Σ converge, since CSD^Σ is $H^{1/2}(\Sigma)$ continuous. Thus, it follows that the limiting configuration has finite energy, and the limiting trajectory it determines on the cylinder is the limit of the sequence of trajectories. Thus, the limit corresponds to a finite energy monopole, and hence $r_\Sigma(M^s)$ is complete.

(ii) For the first part of (ii), similar unique continuation arguments as made in the proof of the main theorem of Part I imply the injectivity of r_Σ and that it is an immersion. To show then that r_Σ is a global embedding, we use similar arguments as made in the proof of Theorem 4.13. Namely, it suffices to show that if $r_\Sigma(\gamma_i)$ forms a Cauchy sequence in $\mathfrak{B}^{s-1/2}(\Sigma)$, then the γ_i form a Cauchy sequence in M^s. However, this follows from our preceding analysis. Namely, we have that the energy of the γ_i converge. On compact cylinders, the γ_i converge in $H^s(I\times\Sigma)$ by Lemma 4.11, and at infinity, we have convergence in a neighborhood of infinity since energy controls exponential decay, i.e., we have equation (6.10). Because of the way M^s is topologized, this gives us convergence of γ_i in M^s.

It remains to prove the more interesting second part of (ii). Let $\mathcal{H}^{s;\pm\delta}_\gamma$ and $\overline{\mathcal{H}}^{s;\pm\delta}_\gamma$ be the Hessian and extended Hessian operators with domains $\mathcal{T}^{s;\pm\delta}$ and $\widetilde{\mathcal{T}}^{s;\pm\delta}$, respectively. Observe that given $[\gamma]\in M^s$, then $r_\Sigma(T_{[\gamma]}M)$ can be regarded as the symplectic reduction of $r_\Sigma(T_\gamma\mathfrak{M}^{s;\delta})$ with respect to the coisotropic subspace $T_\gamma\mu^{-1}(0)$ of $T^{s-1/2}_\Sigma$. We have the following claim:

Claim: The space $r_\Sigma(T_\gamma\mathfrak{M}^{s;\delta})$ is a coisotropic subspace of $\mathcal{T}^{s-1/2}_\Sigma$ with annihilator $r_\Sigma(T_\gamma\mathfrak{M}^{s;\delta}\cap\mathcal{T}^{s;\delta})$.

We will prove this claim, which is equivalent to second assertion of (ii) via symplectic reduction. To prove the first part of the claim, we proceed as follows. First of all, we have

$$\begin{aligned} r_\Sigma(T_\gamma\mathfrak{M}^{s;\delta}) &= r_\Sigma(\ker\widetilde{\mathcal{H}}_\gamma|_{T_\gamma\mathfrak{C}^{s;\delta}(Y)}) \\ &= r_\Sigma(\ker\overline{\mathcal{H}}_\gamma|_{T_\gamma\mathfrak{C}^{s;\delta}(Y)}) \end{aligned} \tag{7.23}$$

where the first equality follows from the definitions and the second follows from Lemma 7.3 and the fact that $\mathcal{J}^{s;\delta}_{\gamma,t}$ has zero restriction to the boundary. Let $\pi_{SR}:\widetilde{\mathcal{T}}^{s-1/2}_\Sigma\to\mathcal{T}^{s-1/2}_\Sigma$ denote the symplectic reduction induced by the coisotropic subspace $W:=\mathcal{T}^{s-1/2}_\Sigma\oplus\Omega^0(\Sigma;i\mathbb{R})\oplus 0$, that is, $\pi_{SR}(x)$ is coordinate projection onto $\mathcal{T}^{s-1/2}_\Sigma$ if $x\in W$ and $\pi_{SR}(x)=0$ otherwise. We will show that

$$r_\Sigma(\ker\overline{\mathcal{H}}_\gamma|_{T_\gamma\mathfrak{C}^{s;\delta}(Y)}) = \pi_{SR}r(\ker\overline{\mathcal{H}}^{s;-\delta/2}_\gamma), \tag{7.24}$$

which together with Lemma 7.4 and (7.23) will show that $r_\Sigma(T_\gamma\mathfrak{M}^{s;\delta})$ is coisotropic.

Let $\mathfrak{a}=\lim_{t\to\infty}\check\gamma(t)$. Then we can write

$$\overline{\mathcal{H}}_\gamma = \overline{\mathcal{H}}_{\gamma_\mathfrak{a}} + R \tag{7.25}$$

where $\overline{\mathcal{H}}_{\gamma_\mathfrak{a}}$ is time-independent and where R is a zeroth order operator whose coefficients belong to $H^{s;\delta}(Y)$. From this, we have

$$\ker\overline{\mathcal{H}}^{s;-\delta/2}_\gamma = \{x\in\widetilde{\mathcal{T}}^{s;\delta/2}+\ker\overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_\mathfrak{a}} : \overline{\mathcal{H}}_\gamma x = 0\}. \tag{7.26}$$

Indeed, if $x\in\widetilde{\mathcal{T}}^{s;-\delta/2}$ and $\overline{\mathcal{H}}_\gamma x=0$, then $\overline{\mathcal{H}}_{\gamma_\mathfrak{a}}x=-Rx\in\mathcal{T}^{s;\delta/2}$. The operator $\overline{\mathcal{H}}_{\gamma_\mathfrak{a}}:\widetilde{\mathcal{T}}^{s+1,\delta/2}\to\widetilde{\mathcal{T}}^{s,\delta/2}$ is surjective (since no boundary conditions are specified), and hence we see that x differs from an element of $\widetilde{\mathcal{T}}^{s+1,\delta/2}$ by an element of $\ker\overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_\mathfrak{a}}$. This proves (7.26).

On the other hand, $\ker\overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_\mathfrak{a}}$ has a simple description, since it is time-independent (it is a cylindrical Atiyah-Patodi-Singer operator, see [3].) For $\delta>0$ sufficiently small so that it is smaller than the the absolute value of the first positive and negative eigenvalue of $\overline{\mathsf{B}}_{\mathfrak{a}}$, we have

$$\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_{\mathfrak{a}}} = \Big\{ \sum_{\lambda \geq 0} c_\lambda \zeta_\lambda e^{-\lambda t} : \Big\| \sum_{\lambda} c_\lambda \zeta_\lambda \Big\|_{H^{s-1/2}(\Sigma)} < \infty \Big\}, \tag{7.27}$$

where the $\{\zeta_\lambda\}_\lambda$ are an orthonormal basis of eigenfunctions for $\overline{\mathsf{B}}_{\mathfrak{a}}$, with ζ_λ having eigenvalue λ. In particular, the space $\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_{\mathfrak{a}}}$ has only at most bounded configurations; none of them are exponentially growing. The ones that are bounded are the time-translation configurations spanned by the kernel of $\overline{\mathsf{B}}_{\mathfrak{a}}$. By Lemma 7.5 however, this is precisely the space (7.19), which is isomorphic to the tangent space to the moduli space of vortices at $[\mathfrak{a}]$. Moreover, the exponentially decaying elements of $\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_{\mathfrak{a}}}$ decay with rate at least $e^{-\delta t/2}$ because of our choice of δ.

Thus, if we let $Z_{\overline{\mathsf{B}}_{\mathfrak{a}}} \subset \ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_{\mathfrak{a}}}$ denote the time-translation invariant elements given by the zero eigenspace of $\overline{\mathsf{B}}_{\mathfrak{a}}$, we have

$$\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma_{\mathfrak{a}}} \subset \widetilde{\mathcal{T}}^{s;\delta/2} + Z_{\overline{\mathsf{B}}_{\mathfrak{a}}}. \tag{7.28}$$

But since $Z_{\overline{\mathsf{B}}_{\mathfrak{a}}} \subset T_{\gamma_{\mathfrak{a}}} \mathscr{V}_{[0,\infty)}$ by Lemma 7.5, equations (7.26) and (7.28) imply that

$$\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma} = \ker \overline{\mathcal{H}}_{\gamma}|_{T_\gamma \mathfrak{C}^{s;\delta/2} \oplus H^{s;\delta/2}(Y)}, \tag{7.29}$$

that is, the only elements of $\overline{\mathcal{H}}^{s;-\delta/2}_{\gamma}$ that do not exponentially decay are those that have a nonzero contribution from $Z_{\overline{\mathsf{B}}_{\mathfrak{a}}} \subset T_{\mathfrak{a}} \mathscr{V}$. Because of the orthogonal decomposition (7.15), elements of $\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma}$ whose restriction under r lie inside the coisotropic space $\mathcal{T}^{s-1/2}_{\Sigma} \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus 0$ have vanishing $H^{s;-\delta/2}(Y)$ component, and thus belong to $T_\gamma \mathfrak{C}^{s;\delta/2}$. (This is exactly the same type of analysis carried out in the symplectic aspects of Section 3.3). This observation together with (7.29) implies

$$\pi_{SR} r(\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma}) = \pi_{SR} r \ker(\overline{\mathcal{H}}_{\gamma}|_{T_\gamma \mathfrak{C}^{s;\delta/2}}).$$

But we have

$$\pi_{SR} r \ker(\overline{\mathcal{H}}_{\gamma}|_{T_\gamma \mathfrak{C}^{s;\delta/2}}) = r_\Sigma \ker(\overline{\mathcal{H}}_{\gamma}|_{T_\gamma \mathfrak{C}^{s;\delta/2}}),$$

and so (7.24) follows from the above two equations. This finishes the first part of the claim.

The second part of the claim is now a simple consequence of Lemma 7.4(ii) and the preceding analysis. Namely, we have that the annihilator of $r_\Sigma(T_\gamma \mathcal{M}^{s;\delta}) = \pi_{SR} r(\ker \overline{\mathcal{H}}^{s;-\delta/2}_{\gamma})$ is given by

$$\begin{aligned} \pi_{SR} r(\ker \overline{\mathcal{H}}^{s;\delta/2}_{\gamma}) &= \pi_{SR} r(\ker \overline{\mathcal{H}}_{\gamma}|_{\mathcal{T}^{s;\delta/2}}) \\ &= r_\Sigma(T_\gamma \mathcal{M}^{s;\delta} \cap T^{s;\delta/2}). \end{aligned}$$

The claim now follows from the fact that $T_\gamma \mathcal{M}^{s;\delta} \cap T^{s;\delta/2}$ modulo gauge is precisely $T_{[\gamma]} M^s_{\partial[\gamma]}$. □

The next theorem considers the space of monopoles $\mathfrak{M}^s_E$ that have small energy less than E, which we study in terms of the corresponding moduli space

$$M_E^s := \{[\gamma] \in M^s : \mathcal{E}(\gamma) < E\}.$$

Geometrically, Theorem 7.6 says that for sufficiently small energy ϵ, the space M_ϵ^s is what we expect it to be in light of the Morse-Bott analysis of the previous section. Namely, M_ϵ^s is an open neighborhood of the the critical set of our flow, the space of vortices, within the stable manifold of the flow. (Since we are working modulo gauge, the stable manifold in question is with respect to the flow on some coordinate patch near a vortex, as we analyzed in the previous section.) Thus, while M_ϵ^s is an infinite-dimensional Hilbert manifold, the only topologically nontrivial portion of it comes from the finite dimensional space of vortices over which it fibers. Furthermore, the Seiberg-Witten flow provides a weak homotopy equivalence from the entire space M^s, whose exact nature we do not know, to the small energy space M_ϵ^s.

Theorem 7.6 *(Small Energy Moduli Space) Let $k = g - 1 - |d| \geq 0$, where g is the genus of Σ.*

(i) For every $E > 0$, the inclusion $M_E^s \hookrightarrow M^s$ induces a weak homotopy equivalence.

(ii) There exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, the space M_ϵ^s is diffeomorphic to a Hilbert ball bundle over the k-vortex moduli space $\mathcal{V}_k(\Sigma)$.

Proof (i) We want to show that the inclusion induces an isomorphism on all homotopy groups. For this, we only have to show that $M_E^s \hookrightarrow M^s$ is surjective on all homotopy groups. So let $f : S_n \to M^s$ be a representative element of $\pi_n(M^s)$ for some n. Observe that for every $T \geq 0$, we have a continuous map $\tau_T : M^s \to M^s$ which translates an element by time T, i.e. $\tau_T(\gamma) = \gamma(\cdot + T)$. Since the image of $f(S_n)$ is compact, and because energy is continuous on M^s, it follows that we can find a large T such that $\tau_T(f(S_n)) \subset M_E^s$. Thus, τ_t, $0 \leq t \leq T$, provides a homotopy from $\tau_T(f(S_n))$ to $f(S_n)$. Since $f : S^n \to M^s$ was arbitrary, this proves the desired surjectivity of the inclusion map on homotopy groups.

(ii) The Chern-Simons-Dirac functional CSD^Σ, being a Morse-Bott functional on the quotient space $\mathfrak{B}_\mu^{s-1/2}(\Sigma)$, is a small lower order perturbation of a positive-definite quadratic form when restricted to small neighborhood of the stable manifold to a critical point. Hence, the level sets of energy on such a stable manifold, for energy close to the energy of the critical set, are just smooth spheres. Thus, the union of those level sets of energy less than ϵ, which is precisely M_ϵ^s, forms a Hilbert ball bundle over $\mathcal{V}$. Here, in this last statement, we implicitly used Lemma 6.1, which tells us that for small enough energy, every configuration is gauge equivalent to a path that remains in a small $H^{1/2}(\Sigma)$ neighborhood of $\mathcal{V}$ for all time, in which case the above local analysis of CSD^Σ near its critical set applies. □

From the previous theorems, we can deduce the following theorem, which allows us to obtain Lagrangian submanifolds of $\mathfrak{B}_\mu^{s-1/2}(\Sigma)$ whose topology we can understand. Namely, we consider the initial data of configurations in $\mathcal{B}_\mu^{s-1/2}(\Sigma)$ that converge under the Seiberg-Witten flow to a submanifold $\mathscr{L}$ inside the vortex moduli space V at infinity. More precisely, define the space

$$M_{\mathscr{L}}^s = \{[\gamma] \in M^s : \partial_\infty[\gamma] \in \mathscr{L}\}.$$

of monopoles in M^s that converge to $\mathscr{L}$. For any $E > 0$, we can also define

$$M^{s;\delta}_{\mathscr{L},E} = M^s_{\mathscr{L}} \cap M^s_E.$$

Theorem 7.7 *Let $\mathscr{L} \subset \mathcal{V}_k(\Sigma)$ denote any Lagrangian submanifold.*

(i) The space $M^s_{\mathscr{L}}$ can be given the topology of a smooth Hilbert manifold. The map $r_\Sigma : M^s_{\mathscr{L}} \to \mathfrak{B}^{s-1/2}_\mu(\Sigma)$ is a diffeomorphism onto a Lagrangian submanifold of $\mathfrak{B}^{s-1/2}_\mu(\Sigma)$. The space $M^s_{\mathscr{L}}$ is weakly homotopy equivalent to a Hilbert ball bundle over $\mathscr{L}$.

(ii) If $\epsilon > 0$ is sufficiently small, then (i) holds with $M^s_{\mathscr{L},\epsilon}$ in place of $M^s_{\mathscr{L}}$, and with "weakly homotopy equivalent" replaced with "diffeomorphic".

Proof Since the map $\partial_\infty : M^s \to \mathcal{V}$ is a smooth submersion, it follows $M^s_{\mathscr{L}} \subset M^s$ has the topology of a smooth Hilbert manifold. From Theorem 7.2(ii), we see that given $[\gamma] \in M^s_{\mathscr{L}}$, the space $r_\Sigma(T_{[\gamma]}M^s_{\mathscr{L}})$ yields a Lagrangian subspace inside the symplectically reduced space

$$r_\Sigma(T_{[\gamma]}M^s)/r_\Sigma(T_{[\gamma]}M^s_{\partial_\infty[\gamma]}).$$

This shows that $r_\Sigma(T_{[\gamma]}M^s_{\mathscr{L}})$ is a Lagrangian subspace of $T_{r_\Sigma[\gamma]}B^{s-1/2}_\mu(\Sigma)$. The remaining statements are now immediate. □

Of course, having worked initially in the Hilbert space setting (as is necessary), one can then restrict to just those configurations that are smooth. Thus, all the results above carry over mutatis mutandis to the smooth setting.

8 The General Case

We conclude Part II with how how one may piece together the results of the previous section on semi-infinite cylinders with the results of Part I to understand the moduli space of monopoles on a general 3-manifold with cylindrical ends. Our main result is Theorem 8.2, which states that after a suitable perturbation, monopole moduli spaces produce immersed Lagrangians in vortex moduli spaces. Moreover, we make some vague remarks about how our work supplies the analysis needed to carry out Donaldson's TQFT formulation of the Seiberg-Witten invariants. The work started here will be completed more fully in the future.

Given a general 3-manifold Y with cylindrical ends, we write it as the union of Y_0, a compact 3-manifold with boundary a disjoint union of Riemann surfaces Σ_i, and cylindrical ends $[0,\infty) \times \Sigma_i$ attached to these boundary components. We suppose that the metric is a product on each of the ends as well as in a tubular neighborhood of ∂Y_0. We suppose that the spin^c structure $\mathfrak{s}$ on Y is a product on the ends such that the determinant line bundle it associates to each Σ_i has Chern-class $d_i \neq 0$. Thus, we have a Morse-Bott situation on each end.

We will define perturbations for our equations which are compactly supported in the interior of a tubular neighborhood of the interface components $\{0\} \times \Sigma_i$ inside Y_0. This is because we want to preserve the cylindrical structure of the equations on the ends. Such a perturbation is given by a function $\mathfrak{q} : \mathfrak{C}(Y_0) \to \mathcal{K}(Y_0)$ which extends to Sobolev completions and which then gives us the modified Seiberg-Witten equations $SW_3(B,\Psi) + \mathfrak{q}(B,\Psi) = 0$ on Y_0. These equations then extend to Y because of the compact support of $\mathfrak{q}$.

Restricting our attention to Y_0 for the moment, we want our perturbations to be compatible with the results developed in Part I for Y_0. One way to ensure this is to only use *local* perturbations, that is, perturbations whose pointwise value at $y \in Y_0$ depends only on the value of (B,Ψ) at y. This is so that the crucial arguments involving unique continuation in Part I continue to hold[8].

The restriction of locality is a strong requirement, but fortunately, there are still a rich enough class of perturbations for us to achieve the desirable transversality results. For notational compactness, we write

$$\Sigma = \coprod_{i=1}^{n} \Sigma_i,$$

so that ∂Y_0 is just the single disconnected surface Σ. Define the open subset $N = (-1,-1/4) \times \Sigma \subset Y_0$ lying in the interior of a tubular neighborhood $(-3/2, 0] \times \Sigma$ of Σ in Y_0. We can consider perturbations of the following form[9].

Let η be an imaginary coclosed 1-form supported in N. This yields for us the perturbation $\mathfrak{q}_\eta(B,\Psi) = \eta$. Next, let U be an open subset of N on which we can trivialize the spinor bundle $\mathcal{S}$. So with respect to some trivialization of $\mathcal{S}|_U \cong \mathbb{C}^2$, we can write the spinor $\Psi = (\Psi_+, \Psi_-)$ as a pair of complex numbers. To obtain a gauge-invariant perturbation, we can take our perturbations to be gradients of gauge-invariant polynomials of the components of Ψ. If we want to obtain a perturbation that is a linear function of the spinor, this limits us to the following quadratic polynomials:

$$q_1(\Psi) = |\Psi_+|^2, \quad q_2(\Psi) = |\Psi_-|^2, \quad q_3(\Psi) = \mathrm{Re}\,(\Psi_+, \Psi_-), \quad q_4(\Psi) = \mathrm{Im}\,(\Psi_+, \Psi_-).$$

One can check that at any point of Y for which Ψ is nonzero, the gradients of these four functions (with respect to the real inner product $\mathrm{Re}\,(\cdot,\cdot)$ on $\mathcal{S}$) span the orthogonal complement to the vector $i\Psi$ spanning the infinitesimal gauge orbit of Ψ. Thus, given a quadruple of smooth real valued functions $\vec{f} = (f_i)_{i=1}^4$ compactly supported in U, we obtain a gauge invariant perturbation by defining $\mathfrak{q}_{\vec{f}}$ to be the gradient of $\sum f_i q_i$.

We will use the above two types of perturbations as our main building blocks. Namely, we can now proceed to construct a large Banach space of perturbations, in the sense of Kronheimer-Mrowka (see [21, Chapter 11.6]). We fix the following data:

- a finite open cover $\{U_i\}_{i=1}^m$ of $(-3/4, -1/2) \times \Sigma$, where $\bar{U}_i \subset N$, along with trivializa-

[8] The general class of perturbations used in [21] on a 3-manifold, obtained from cylinder functions, do not satisfy this property for the 3-dimensional Seiberg-Witten equations. However, they do satisfy a locality property for the corresponding 4-dimensional Seiberg-Witten equations on a cylindrical 4-manifold $\mathbb{R} \times Y$ (at every time $t \in \mathbb{R}$, the value of the perturbation depends only on the configuration at that particular time). Thus, the perturbations of [21] are suited for the 4-dimensional Seiberg-Witten equations, but not for the 3-dimensional equations on a manifold with boundary. See [5] for further reading on the relationship between unique continuation and locality.

[9] One could work with the cylinder function perturbations of [21] by working in an S^1 invariant setting. Namely, consider S^1-invariant cylinder functions on $S^1 \times \Sigma_i$ and proceed as in [21] to get perturbations defined in a "slicewise" fashion compactly supported in the product neighborhood N. They will be local in with respect to the $t \in (-3/2, 0]$ variable, which is sufficient. The perturbations we describe are an alternate set of perturbations, which are more simple, albeit perhaps a bit crude since it requires some ad hoc choices of local bundle trivializations.

tions of $\mathcal{S}|_{U_j}$;

- a countable collection of smooth coclosed imaginary 1-forms η_i compactly supported on some U_j;

- a countable collection of quadruples $\vec{f_i} = (f_{i,1}, f_{i,2}, f_{i,3}, f_{i,4})$ of smooth real-valued functions compactly supported in some U_j,

where the countable collection of elements are chosen to be dense in the $C_0^\infty(\cup_i U_i)$ topology in the space of all such data. We then obtain a large Banach space $\mathcal{P}$ of perturbations which is spanned by all finite linear combinations $\sum(\lambda_i \mathfrak{q}_{\eta_i} + \lambda_i' \mathfrak{q}_{\vec{f_{i'}}}$ of the perturbations generated as above, $\lambda_i, \lambda_i' \in \mathbb{R}$.

We now state our main theorem below and sketch its proof. First, we introduce some notation. Given a perturbation $\mathfrak{q} \in \mathcal{P}$, we can consider the $\mathfrak{q}$-perturbed Seiberg-Witten equations

$$SW_{3,\mathfrak{q}}(B,\Psi) := SW_3(B,\Psi) + \mathfrak{q}(B,\Psi) = 0$$

on Y_0 and on Y. Consider the corresponding moduli space of all smooth monopoles on Y_0 and Y, where on Y, we require that the energy be finite on the ends:

$$M_{\mathfrak{q}}(Y_0) = \{\gamma \in \mathfrak{C}(Y_0) : SW_{3,\mathfrak{q}}(\gamma) = 0\}/\mathcal{G}(Y_0) \tag{8.1}$$

$$M_{\mathfrak{q}}(Y) = \{\gamma \in \mathfrak{C}(Y) : SW_{3,\mathfrak{q}}(\gamma) = 0,\ \mathcal{E}(\gamma|_{[0,\infty)\times\Sigma_i}) < \infty,\ 1 \le i \le n\}/\mathcal{G}(Y). \tag{8.2}$$

Here, $M_{\mathfrak{q}}(Y)$ is topologized via $C^\infty_{\mathrm{loc}}(Y)$ and via the requirement that the energy functional on the ends be continuous. Note that since Y no longer has a boundary, every Sobolev monopole on Y is gauge equivalent to a smooth one, which is why we omitted Sobolev completions in the above quotient.

As in [21], we can describe $\mathcal{M}_{\mathfrak{q}}(Y)$ as a fiber product of the moduli space of monopoles on Y_0 and on the ends. We have restriction maps to the quotient configuration space $\mathfrak{B}(\Sigma)$ on the interface $\Sigma = \{0\} \times \Sigma$ of Y_0 and $[0,\infty) \times \Sigma$:

$$r_\Sigma^+ : \mathfrak{B}(Y_0) \to \mathfrak{B}(\Sigma) \tag{8.3}$$

$$r_\Sigma^- : \mathfrak{B}([0,\infty) \times \Sigma) \to \mathfrak{B}(\Sigma). \tag{8.4}$$

Via restriction, these maps then give us maps

$$r_\Sigma^+ : M_{\mathfrak{q}}(Y_0) \to \mathfrak{B}(\Sigma) \tag{8.5}$$

$$r_\Sigma^- : M([0,\infty) \times \Sigma) \to \mathfrak{B}(\Sigma). \tag{8.6}$$

One can show, as in [21, Lemma 24.2.2], the following:

Lemma 8.1 *The natural map $M_{\mathfrak{q}}(Y) \to M_{\mathfrak{q}}(Y_0) \times M([0,\infty) \times \Sigma)$ yields a homeomorphism from $M_{\mathfrak{q}}(Y)$ onto the fiber product of (8.5) and (8.6).*

Our main result is the following. Pick a spin^c structure $\mathfrak{s}$ on Y as above, and let $k_i = g_i - 1 - \frac{|d_i|}{2}$, where g_i is the genus of Σ_i. The ends Σ_i of Y yield for us the product of vortex moduli spaces $\prod_{i=1}^n \mathcal{V}_{k_i}(\Sigma_i)$ endowed with the product symplectic structure. We can define the smooth map

$$\partial_\infty : M_{\mathfrak{q}}(Y) \to \prod_{i=1}^{n} \mathcal{V}_{k_i}(\Sigma_i) \tag{8.7}$$

which sends a monopole to the gauge-equivalence class of its limit on each end.

Theorem 8.2 *For a residual set of perturbations $\mathfrak{q} \in \mathcal{P}$, the space $M_{\mathfrak{q}}(Y)$ is a smooth, compact manifold. Moreover, the map (8.7) is a Lagrangian immersion.*

Proof From the fiber product description of Lemma 8.1, one can argue similarly as in Proposition 24.3.2 and Lemma 24.4.8 to achieve transversality for the map (8.5) and (8.6) for a residual set of perturbations. Though our class of perturbations is different, the same type of arguments carry through, since away from reducible configurations (which we do not have to worry about due to our choice of $\mathfrak{s}$), our perturbations are sufficiently rich (in the interior of N, they yield vector fields that are dense in the orthogonal complement to the action of the gauge group). Thus, by achieving transversality, we have that $M_{\mathfrak{q}}(Y)$ is a smooth manifold.

The fact that $\mathcal{M}_{\mathfrak{q}}(Y)$ is compact follows from the compactness results for the perturbed Seiberg-Witten equations, see [21, Chapter 24.5]. In our situation, all finite energy configurations in $\mathcal{M}_{\mathfrak{q}}(Y)$ must have the exactly the same (perturbed) topological energy, since the space of vortices on each end is connected (and so CSD^{Σ_i} has constant value on the vortices on each Σ_i). Moreover, we cannot have trajectory breaking on the ends for the same reason: the only finite energy solutions on an infinite cylinder $(-\infty, \infty) \times \Sigma_i$ are translation-invariant zero energy vortices. Thus, our space $\mathcal{M}_{\mathfrak{q}}(Y)$ is compact as is.

For the second statement, we can see this very easily in geometric terms. From Part I, we know that the image of (8.5) is a Lagrangian submanifold. (Our perturbations were carefully chosen so that the results of Part I still apply. Indeed, they are linear, local, and supported away from the boundary, and so one can check that this does not affect the analysis of Part I.) Let $[\gamma] \in M([0,\infty) \times \Sigma)$ and define $[\mathfrak{a}] := \partial_\infty[\gamma] \in \prod_{i=1}^n \mathcal{V}_{k_i}(\Sigma_i)$. Note that the differential of ∂_∞ at $[\gamma] \in M([0,\infty) \times \Sigma)$ has kernel precisely equal to $T_{[\gamma]}M_{[\mathfrak{a}]}([0,\infty) \times \Sigma)$, the tangent space to the stable manifold to $[\mathfrak{a}]$. On the other hand, by Theorem 7.2(ii), we have that $r_\Sigma^-\big(T_{[\gamma]}M_{[\mathfrak{a}]}([0,\infty) \times \Sigma)\big)$ is an isotropic subspace annihilating the coisotropic subspace $r_\Sigma^-\big(T_{[\gamma]}M([0,\infty) \times \Sigma)\big)$. So given any $u \in M_{\mathfrak{q}}(Y)$, it follows that the differential

$$D_u\partial_\infty : T_uM_{\mathfrak{q}}(Y) \to T_{\partial_\infty[u]}\left(\prod_{i=1}^{n} \mathcal{V}_{k_i}(\Sigma_i)\right)$$

has range isomorphic to the symplectic reduction of the Lagrangian subspace $T_{u|_\Sigma}\big(r_\Sigma^+ M_q(Y_0)\big)$ coming from Y_0 with respect to the coisotropic space $T_{u|_\Sigma} r_\Sigma^-\big(M([0,\infty) \times \Sigma)\big)$ coming from the ends. In particular, the differential of ∂_∞ at any monopole on Y has image a Lagrangian subspace. Moreover, the map ∂_∞ is an immersion due to the transversality of the maps (8.5) and (8.6). This proves the theorem. □

We conclude by noting that because $\mathcal{M}_{\mathfrak{q}}(Y)$ is compact, it has a fundamental class with which we may execute push-pull maps on the homology of the vortex moduli spaces

$\Pi_{i=1}^n \mathcal{V}_i(\Sigma)$ on the ends. In particular, the TQFT invariant that Donaldson studies in [10] may now be actually realized as a signed count of solutions to the monopole equations on a closed 3-manifold. Indeed, suppose we have a closed 3-manifold $\bar{Y}$. If we remove a separating hypersurface Σ and stretch the neck to infinity, we now have a new 3-manifold Y with two ends modeled on $[0,\infty) \times \Sigma$. A signed count of the solutions to the (perturbed) Seiberg-Witten equations on $\bar{Y}$ should then correspond to a signed intersection number of the image of (8.7) with the diagonal inside $\mathcal{V}(\Sigma) \times \mathcal{V}(\Sigma)$. For this, one can show (with very little extra work) that for a residual set of perturbations $\mathfrak{q}$, the image of (8.7) is transverse to the diagonal. We will discuss this more fully in future work.

Part III

The Seiberg-Witten Equations with Lagrangian Boundary Conditions

9 Introduction

Consider the Seiberg-Witten equations on a 4-manifold X. These equations are a system of nonlinear partial differential equations for a connection and spinor on X. When X is a product $\mathbb{R} \times Y$, where Y is a closed 3-manifold, the Seiberg-Witten equations on $\mathbb{R} \times Y$ become the formal downward gradient flow of the Chern-Simons-Dirac functional on Y. The associated Floer theory of the Chern-Simons-Dirac functional has been extensively studied, and after setting up the appropriate structures, we obtain the monopole Floer homology groups of Y, which are interesting topological invariants of Y (see [21]).

In Part III, we consider the case when Y is a 3-manifold with boundary. To obtain well-posed equations on $\mathbb{R} \times Y$ in this case, we must impose boundary conditions for the Seiberg-Witten equations. Following the approach of [54] and [42], we impose Lagrangian boundary conditions, which means that at every time $t \in \mathbb{R}$, a solution of our equations must have its boundary value lying in a fixed Lagrangian submanifold $\mathfrak{L}$ of the boundary configuration space. The resulting equations become a Floer type equation on the space of configurations whose boundary values lie in $\mathfrak{L}$. Understanding the analytic underpinnings of the Seiberg-Witten equations on $\mathbb{R} \times Y$ with Lagrangian boundary conditions is therefore a first step in defining a monopole Floer theory for the pair $(Y, \mathfrak{L})$ of a 3-manifold Y with boundary and a Lagrangian $\mathfrak{L}$.

Part III is the analogue of [54] for the Seiberg-Witten setting, since [54] establishes similar foundational analytical results for the anti-self-dual (ASD) equations with Lagrangian boundary conditions. The analysis there was eventually used to construct an instantion Floer homology with Lagrangian boundary conditions in [42]. This latter work was the first to construct a gauge-theoretic Floer theory using Lagrangian boundary conditions. The original motivation for [42] was to prove the Atiyah-Floer conjecture in the ASD setting (see also [56]). Informally, this conjecture states the following: given a homology 3-sphere Y with a Heegard splitting $H_0 \cup_\Sigma H_1$, where H_0 and H_1 are two handlebodies joined along the surface Σ, there should be a natural isomorphism between the instanton Floer homology for Y and the symplectic Floer homology for the pair of Lagrangians $(\mathcal{L}_{H_0}, \mathcal{L}_{H_1})$ inside the representation variety of Σ. Here, $\mathcal{L}_{H_i}$ is the moduli space of flat connections on Σ that extend to H_i, $i = 0, 1$. As explained in [56], instanton Floer homology with Lagrangian boundary conditions is expected to serve as an intermediary Floer homology theory in proving the Atiyah-Floer conjecture, where the instanton Floer homology of the pair $([0,1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$ should interpolate between the two previous Floer theories.

For the Seiberg-Witten setting, one could also formulate an analogous Atiyah-Floer type conjecture, although in this case, one ends up with infinite-dimensional Lagrangians inside an infinite-dimensional symplectic quotient (we will discuss this more thoroughly later). One could also expect (as in the instanton case) that a monopole Floer homology for a 3-manifold Y with boundary Σ supplied with suitable Lagrangian boundary conditions should recover the usual monopole Floer homology for closed extensions $\bar{Y} = Y \cup_\Sigma Y'$ of Y, where the bounding 3-manifolds Y' satisfy appropriate hypotheses. These considerations served as our preliminary motivation for laying the foundational analysis for a monopole Floer homology with Lagrangian boundary conditions. Recently, several other Floer theories on 3-manifolds with boundary have been constructed, in particular, the bordered Heegaard Floer homology theory of Lipshitz-Ozsváth-Thurston [26] and the sutured monopole Floer homology theory of Kronheimer-Mrowka [22]. A complete construction of a monopole Floer theory with Lagrangian boundary conditions would therefore add to this growing list of Floer theories, and it would be of interest to understand what relationships, if any, exist among all these theories.

Basic Setup and Main Results

In order to give precise meaning to the notion of a Lagrangian boundary condition for the Seiberg-Witten equations, we first explain the infinite-dimensional symplectic aspects of our problem. We will then explain the geometric significance of our setup and its applicability to Floer homology after a precise statement of our main results. Recall from Part I that the boundary configuration space

$$\mathfrak{C}(\Sigma) = \mathcal{A}(\Sigma) \times \Gamma(\mathcal{S}_\Sigma)$$

of connections and spinors on the boundary comes equipped with the symplectic form

$$\omega((a,\phi),(b,\psi)) = \int_\Sigma a \wedge b + \int_\Sigma \mathrm{Re}\,(\phi, \rho(\nu)\psi), \qquad (a,\phi),(b,\psi) \in \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma). \tag{9.1}$$

on each of its tangent spaces. Here, $\mathcal{S}_\Sigma$ is the spinor bundle on $\mathbb{R} \times Y$ restricted to Σ and $\rho(\nu)$ is Clifford multiplication by the outward normal ν to Σ. The form ω is symplectic because it has a compatible complex structure

$$J_\Sigma = (-\check{*}, -\rho(\nu)), \tag{9.2}$$

that is, $\omega(\cdot, J_\Sigma \cdot)$ is the L^2 inner product on $\mathcal{T}_\Sigma := \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma)$ naturally induced from the Riemannin metric on differential forms and the real part of the Hermitian inner product on the space of spinors. It follows that the L^2 closure of the configuration space $L^2\mathfrak{C}(\Sigma)$ is a Hilbert manifold whose tangent spaces $L^2\mathcal{T}_\Sigma$ are all strongly symplectic Hilbert spaces (see Section 19).

It follows that ω induces a nondegenerate skew-symmetric form on any topological vector space $\mathcal{X}$ densely contained in $L^2\mathcal{T}_\Sigma$. As in Section 19, we still call $\omega|_\mathcal{X}$ a symplectic form on $\mathcal{X}$. We need to consider the restriction of ω to other topological vector spaces not only because we work with the smooth configuration space. Since we will be considering Sobolev spaces on $\mathbb{R}\times Y$, we also need to complete the boundary configuration space in Besov spaces, these latter spaces being boundary value spaces of Sobolev spaces. More precisely, given a smooth manifold M (possibly with boundary), we can consider the Sobolev spaces $H^{s,p}(M)$ and Besov spaces $B^{s,p}(M)$ on M, where $s \in \mathbb{R}$ and $1 < p < \infty$. When s is a nonnegative integer, $H^{s,p}(M)$ is just the usual space of functions that have all derivatives up to order s belonging to $L^p(M)$. The Besov spaces are defined as in Part IV, and their most important feature is that for any $s > k/p$, if $N \subset M$ is a codimension k submanifold of M, there is a continuous restriction map

$$\begin{aligned} r_N : H^{s,p}(M) &\to B^{s-k/p,p}(N) \\ f &\mapsto f|_N. \end{aligned} \tag{9.3}$$

Thus, restriction to a submanifold maps a Sobolev space on M into a Besov space on N and decreases the order of regularity by k/p. (Note that for $p = 2$, Besov spaces coincide with Sobolev spaces and the above result becomes the familiar fact that a codimension k restriction decreases regularity by $k/2$ fractional derivatives).

For $s > 0$ and $p \geq 2$, we have an inclusion $B^{s,p}(\Sigma) \subset L^2(\Sigma)$. Thus, consider $B^{s,p}\mathfrak{C}(\Sigma)$, the closure the smooth configuration space $\mathfrak{C}(\Sigma)$ in the $B^{s,p}(\Sigma)$ topology. The symplectic form (9.1) induces a (weak) symplectic form on the Banach configuration space $B^{s,p}\mathfrak{C}(\Sigma)$ and the smooth configuration space $\mathfrak{C}(\Sigma)$. Since ω possesses a compatible complex structure J_Σ, we can define a Lagrangian subspace of $\mathcal{T}_\Sigma$ to be a closed subspace L such that $\mathcal{T}_\Sigma = L \oplus J_\Sigma L$ as a direct sum of topological vector spaces (see Section 19 for further reading). A Lagrangian submanifold of $\mathfrak{C}(\Sigma)$ is then a (Fréchet) submanifold of $\mathfrak{C}(\Sigma)$ for which each tangent space is a Lagrangian subspace of $\mathcal{T}_\Sigma$. A Lagrangian subspace (submanifold) of the $B^{s,p}(\Sigma)$ completion of $\mathcal{T}_\Sigma$ is defined similarly.

Definition 9.1 Fix $p > 2$. An *$H^{1,p}$ Lagrangian boundary condition* is a choice of a closed Lagrangian submanifold $\mathfrak{L}$ of $\mathfrak{C}(\Sigma)$ whose closure $\mathfrak{L}^{1-2/p,p}$ in the $B^{1-2/p,p}(\Sigma)$ topology is a smoothly embedded Lagrangian submanifold of $B^{1-2/p,p}\mathfrak{C}(\Sigma)$.

Here, the modifier $H^{1,p}$, which we may omit in the future for brevity, expresses the fact that we will be considering our Seiberg-Witten equations on $\mathbb{R}\times Y$ in the $H^{1,p}_{\mathrm{loc}}(\mathbb{R}\times Y)$ topology. Here, $H^{s,p}_{\mathrm{loc}}(\mathbb{R}\times Y)$, denotes the space of functions whose restriction to any compact subset $K \subset \mathbb{R}\times Y$ belongs to $H^{s,p}(K)$, $s \in \mathbb{R}$.

The significance of a Lagrangian boundary condition is that we can impose the following boundary conditions for a spin^c connection A and spinor Φ on $\mathbb{R}\times Y$ of regularity $H^{1,p}_{\mathrm{loc}}(\mathbb{R}\times Y)$. Namely, we require

$$(A,\Phi)|_{\{t\}\times\Sigma} \in \mathfrak{L}^{1-2/p,p}, \qquad \forall t \in \mathbb{R}, \tag{9.4}$$

i.e., the restriction of (A,Φ) to every time-slice $\{t\}\times\Sigma$ of the boundary lies in the Lagrangian submanifold $\mathfrak{L}^{1-2/p,p}$. Here, we made use of (9.3) with $k = 2$. Note that this restriction theorem requires $p > 2$ when $s = 1$, thereby requiring $p > 2$ Sobolev spaces for the analysis on $\mathbb{R} \times Y$ and the subsequent use of Besov space on the boundary. We require our boundary condition to be given by a Lagrangian submanifold because it allows us to give a Morse-Novikov-Floer theoretic interpretation of the Seiberg-Witten equations supplied with the boundary conditions (9.4), a viewpoint which we discuss more thoroughly after stating our main results[1].

From the results of Part I, we have a natural class of Lagrangian boundary conditions. Namely, consider a 3-manifold Y' with $\partial Y' = -\Sigma$ and such that $Y' \cup_\Sigma Y$ is a smooth Riemannian 3-manifold. Moreover, suppose the spin^c structure $\mathfrak{s}$ on Y extends smoothly to a spin^c structure $\mathfrak{s}'$ over Y'. In such a case, the boundary configuration spaces arising from Y and Y' can be identified, and so can their Lagrangian submanifolds, since the symplectic forms induced on $\mathfrak{C}(\Sigma)$ from Y' and Y differ by a minus sign[2] (the induced orientation on Σ differ in the two cases).

Consequently, the main theorem of Part I provides us with a Lagrangian boundary condition. Namely, define

$$\mathcal{L}(Y', \mathfrak{s}') := \{(B', \Psi')|_\Sigma : (B', \Psi') \in \mathfrak{C}(Y'),\ SW_3(B', \Psi') = 0\} \subset \mathfrak{C}(\Sigma) \tag{9.5}$$

to be the space of boundary values of connections and spinors (B', Ψ') belonging to the configuration space $\mathfrak{C}(Y')$ on Y' that solve the monopole equations $SW_3(B', \Psi') = 0$ on Y'. Then if

$$c_1(\mathfrak{s}') \text{ is non-torsion or } H^1(Y', \Sigma) = 0 \tag{9.6}$$

the main result of Part I is that $\mathcal{L}(Y', \mathfrak{s}')$ is an $H^{1,p}$ Lagrangian boundary condition for $p > 4$.

Definition 9.2 Let Y' and $\mathfrak{s}'$ be as above, with $\mathfrak{s}'$ satisfying (9.6). Then we call the Lagrangian submanifold $\mathcal{L}(Y', \mathfrak{s}') \subset \mathfrak{C}(\Sigma)$ a *monopole Lagrangian.*

Our main result is that the Seiberg-Witten equations $SW_4(A, \Phi) = 0$, defined by (9.14), supplied with a Lagrangian boundary condition arising from a monopole Lagrangian yields an elliptic boundary value problem, i.e., one for which elliptic regularity modulo gauge holds. Here the gauge group is $\mathcal{G} = \mathrm{Maps}(\mathbb{R} \times Y, S^1)$ and it acts on the configuration space $\mathfrak{C}(\mathbb{R} \times Y)$ of spin^c connections and spinors on $\mathbb{R} \times Y$ (where the spin^c structure on $\mathbb{R} \times Y$ has been fixed and pulled back from a spin^c structure on Y) via

$$(A, \Phi) \mapsto g^*(A, \Phi) = (A - g^{-1}dg, g\Phi).$$

Let $\mathcal{G}_{\mathrm{id}}$ denote the identity component of the gauge group, let the prefix $H^{s,p}_{\mathrm{loc}}$ denote closure with respect to the $H^{s,p}_{\mathrm{loc}}(\mathbb{R} \times Y)$ topology.

Theorem A (Regularity). *Let $p > 4$, and let $(A, \Phi) \in H^{1,p}_{\mathrm{loc}}\mathfrak{C}(\mathbb{R} \times Y)$ solve the boundary*

[1] The Lagrangian property is also crucial for the analytic details of the proofs of our main results, see the outline at the end of this introduction.

[2] Since this is the essential property for Y', in actuality, one merely need that the Riemannian metric on $Y' \cup_\Sigma Y$ be continuous instead of smooth.

value problem

$$\begin{aligned} SW_4(A,\Phi) &= 0 \\ (A,\Phi)|_{\{t\}\times\Sigma} &\in \mathfrak{L}^{1-2/p,p}, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{9.7}$$

where $\mathfrak{L}^{1-2/p,p}$ denotes the $B^{1-2/p,p}(\Sigma)$ closure of a monopole Lagrangian $\mathfrak{L}$. Then there exists a gauge transformation $g \in H^{2,p}_{\mathrm{loc}}\mathcal{G}_{\mathrm{id}}$ such that $g^*(A,\Phi)$ is smooth.

Next, we have a compactness result for sequences of solutions provided that the Lagrangian $\mathfrak{L}$ is invariant under the gauge group action of $\mathcal{G}(Y)|_\Sigma$. If $\mathfrak{L}$ satisfies this, we say that $\mathfrak{L}$ is *fully gauge-invariant.* Observe that if we take $Y' = -Y$ and $\mathfrak{s}'$ the spin^c structure on Y, then $\mathcal{L}(Y',\mathfrak{s}')$ will be a fully gauge-invariant monopole Lagrangian, provided Y' and $\mathfrak{s}'$ satisfy (9.6). In general, the condition that $\mathcal{L}(Y',\mathfrak{s}')$ be fully gauge-invariant is precisely the condition that the natural restriction maps on cohomology $H^1(Y) \to H^1(\Sigma)$ and $H^1(Y') \to H^1(\Sigma)$ have equal images in $H^1(\Sigma)$. In this situation, the following theorem says that if we have a local bound on the $H^{1,p}$ "energy" of a sequence of configurations (A_i,Φ_i), then modulo gauge, a subsequence converges smoothly on every compact subset of $\mathbb{R}\times Y$. Here, the energy of a configuration on a compact set K is given by the gauge-invariant norms appearing in (9.8), where ∇_A denotes the spin^c covariant derivative determined by the connection A.

Theorem B (Compactness). *Let $p > 4$ and let $(A_i,\Phi_i) \in H^{1,p}_{\mathrm{loc}}\mathfrak{C}(\mathbb{R}\times Y)$ be a sequence of solutions to (9.7), where $\mathfrak{L}$ is a fully gauge-invariant monopole Lagrangian. Suppose that on every compact subset $K \subset \mathbb{R}\times Y$, we have*

$$\sup_i \|F_{A_i}\|_{L^p(K)}, \|\nabla_{A_i}\Phi_i\|_{L^p(K)}, \|\Phi_i\|_{L^p(K)} < \infty. \tag{9.8}$$

Then there exists a subsequence of configurations, again denoted by (A_i,Φ_i), and a sequence of gauge transformations $g_i \in H^{2,p}_{\mathrm{loc}}\mathcal{G}$ such that $g_i^*(A_i,\Phi_i)$ converges in $C^\infty(K)$ for every compact subset $K \subset \mathbb{R}\times Y$.

Both Theorems A and B apply verbatim to the periodic setting, where $\mathbb{R}\times Y$ is replaced with $S^1\times Y$, in which case, we can work with $H^{k,p}(S^1\times Y)$ spaces instead of $H^{k,p}_{\mathrm{loc}}(\mathbb{R}\times Y)$. In fact we will prove Theorems A and B in the periodic setting (where $\mathbb{R}\times Y$ is replaced with $S^1\times Y$), which then implies the result on $\mathbb{R}\times Y$ by standard patching arguments. In the periodic setting we will also prove that the linearization of (9.7) is Fredholm in a suitable gauge (and in suitable topologies), see Theorem 11.8. One can also prove the Fredholm property on $\mathbb{R}\times Y$ assuming suitable decay hypotheses at the ends, but we will not pursue that here.

Note that the requirement $p > 4$ in the above is sharp with respect to the results in Part I, in that for no value of $p \le 4$ is it known that $\mathfrak{L}^{1-2/p,p}$ is a smooth Banach manifold. Thus, our results here cannot be sharpened unless the results in Part I are also sharpened. On the other hand, the value $p < 4$ is a priori unsatisfactory from the point of view of Floer theory. This is because the a priori energy bounds we have on solutions to the Seiberg-Witten equations, namely the analytic and topological energy as defined in [21], are essentially an $H^{1,2}$ control. Therefore, Theorem B is not sufficient to guarantee compactness results for the moduli space of solutions to (9.7) that are of the type needed for a Floer theory. However, this is not the end the story, as can be seen in the ASD situation, where a Floer theory still exists even though the analogous regularity and compactness results are proven only for $p > 2$, which although better than $p > 4$, still misses $p = 2$.[3] The ASD Floer theory is possible due to the bubbling analysis carried out in [55] and the presence of energy-index formulas in [42], which allow one to use the $p > 2$ analysis to understand the compactification of the space Floer trajectories between critical points. We will leave the study of the analog of such issues in the Seiberg-Witten setting for the future, namely, the study of what can happen to a sequence of solutions to (9.7) if one is only given an $H^{1,2}$ type energy bound (more precisely, a bound on the analytic and topological energy of [21]). At present then, our main theorems therefore serve the foundational purpose of showing that the Seiberg-Witten equations with Lagrangian boundary conditions are well-posed and satisfy a weak type of compactness. These results are key for a future construction of an associated Floer theory.

Geometric Origins

Having stated our main results, we explain how the Seiberg-Witten equations supplied with Lagrangian boundary conditions naturally arise in trying to construct a Floer homology on a 3-manifold with boundary. On a product $\mathbb{R} \times Y$, for Y with or without boundary, the Seiberg-Witten equations take the following form.

We have a decomposition

$$\mathcal{A}(\mathbb{R} \times Y) = \mathrm{Maps}(\mathbb{R}, \mathcal{A}(Y)) \times \mathrm{Maps}(\mathbb{R}, \Omega^0(Y; i\mathbb{R})) \tag{9.9}$$

whereby a connection $A \in \mathcal{A}(\mathbb{R} \times Y)$ on $\mathbb{R} \times Y$ can be decomposed as

$$A = B(t) + \alpha(t)dt, \tag{9.10}$$

where $B(t) \in \mathcal{A}(Y)$ is a path of connections on Y and $\alpha(t) \in \Omega^0(Y; i\mathbb{R})$ is a path of 0-forms on Y, $t \in \mathbb{R}$. Likewise, if we write $\mathcal{S}^+$ for the bundle of self-dual spinors on Y and write $\mathcal{S}$ for the spinor bundle on Y obtained by restriction of $\mathcal{S}^+$ to $\{0\} \times Y$, we can write

$$\mathcal{S}^+ = \mathrm{Maps}(\mathbb{R}, \mathcal{S}), \tag{9.11}$$

where we have identified $\mathcal{S}^+$ with the pullback of $\mathcal{S}$ under the natural projection of $\mathbb{R} \times Y$ onto Y. Thus, any spinor Φ on X is given by a path $\Phi(t)$ of spinors on Y. Altogether, the configuration space

$$\mathfrak{C}(\mathbb{R} \times Y) = \mathcal{A}(\mathbb{R} \times Y) \times \Gamma(\mathcal{S}^+)$$

[3] In the ASD situation, it is possible to prove that $L^p\mathfrak{L}$, the L^p closure of $\mathfrak{L}$, is a smooth submanifold of the space of L^p connections, $p > 2$. Since we have the embedding $B^{1-2/p,p}(\Sigma) \hookrightarrow L^p(\Sigma)$, the ASD equations are well-behaved for $p > 2$, and thus the analogue of Theorem A with $p > 2$ is the optimal result there. Note however, in both the ASD and Seiberg-Witten setting, $p = 2$ can never be achieved, since then a Lagrangian boundary condition cannot be defined. Indeed, a function belonging to $H^{1,2}$ does not have a well-defined restriction to a codimension two submanifold.

on $\mathbb{R} \times Y$ can be expressed as a configuration space of paths:

$$\begin{aligned} \mathfrak{C}(\mathbb{R} \times Y) &= \mathrm{Maps}(\mathbb{R}, \mathcal{A}(Y) \times \Gamma(\mathcal{S}) \times \Omega^0(Y; i\mathbb{R})) \\ (A, \Phi) &\mapsto (B(t), \Phi(t), \alpha(t)). \end{aligned} \tag{9.12}$$

Under this correspondence, the Seiberg-Witten equations can be written as follows (see [21, Chapter 4]):

$$SW_4(A, \Phi) := \left(\frac{d}{dt} B + \left(\frac{1}{2} *_Y F_{B^t} + \rho^{-1}(\Phi\Phi^*)_0 \right) - d\alpha, \frac{d}{dt}\Phi + D_B\Psi + \alpha\Phi \right) \tag{9.13}$$

$$= 0, \tag{9.14}$$

where we have suppressed the time-dependence from the notation. Here, the terms appearing on the right-hand-side of (9.13) are defined as in Part I. In particular, when $\alpha \equiv 0$, i.e. when A is in temporal gauge, then the Seiberg-Witten equations (9.14) are equivalent to the equations

$$\begin{aligned} \frac{d}{dt} B &= -\left(\frac{1}{2} *_Y F_{B^t} + \rho^{-1}(\Phi\Phi^*)_0 \right) \\ \frac{d}{dt}\Phi &= -D_B\Phi. \end{aligned} \tag{9.15}$$

For $(B, \Psi) \in \mathcal{A}(Y) \times \Gamma(\mathcal{S})$, we have the three-dimensional Seiberg-Witten map

$$SW_3(B, \Psi) := \left(\frac{1}{2} *_Y F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0, D_B\Psi \right) \in \Omega^1(Y; i\mathbb{R}) \times \Gamma(\mathcal{S}), \tag{9.16}$$

which we may think of as a vector field on

$$\mathfrak{C}(Y) = \mathcal{A}(Y) \times \Gamma(\mathcal{S}),$$

the configuration space on Y, since $T_{(B,\Psi)}\mathfrak{C}(Y) = \Omega^1(Y; i\mathbb{R}) \times \Gamma(\mathcal{S})$. Thus, (9.15) is formally the downward "flow" of SW_3. Moreover, the equations (9.14) and (9.15) are equivalent modulo gauge transformations.

When Y is closed, the vector field SW_3 is the gradient of a functional, the *Chern-Simons-Dirac functional* $CSD : \mathfrak{C}(Y) \to \mathbb{R}$. This functional is defined by

$$CSD(B, \Psi) = -\frac{1}{2} \int_Y (B - B_0) \wedge (F_B + F_{B_0}) + \frac{1}{2} \int_Y (D_B\Psi, \Psi),$$

where B_0 is any fixed reference connection. Thus, since $SW_3(B, \Psi)$ is the gradient of CSD at (B, Ψ), (9.15) is formally the downward gradient flow of CSD. In this way, the monopole Floer theory of Y can be thought of informally as the Morse theory of the Chern-Simons-Dirac functional. However, we ultimately must work modulo gauge, and in doing so, the Chern-Simons-Dirac functional does not descend to a well-defined function on the quotient configuration space, but it instead becomes an S^1 valued function. This is because the first term of CSD is a Chern-Simons term which is not fully gauge-invariant but which changes by an amount depending on the homotopy class of the gauge transformation. In this case, when we take the differential of CSD as a circle-valued function on the quotient configuration space[4], we obtain not an exact form but a closed form. On compact manifolds, the Morse theory for a closed form is more generally known as Morse-Novikov theory, from which the case of exact forms reduces to the usual Morse theory. Thus, the monopole Floer theory of Y is more accurately the Morse-Novikov theory for the differential of CSD on the quotient configuration space.

It is this point of view which we wish to adopt in trying to generalize monopole Floer theory to manifolds with boundary. Suppose $\partial Y = \Sigma$ is nonempty. We now impose boundary conditions for (9.15) that preserve the above Morse-Novikov viewpoint for monopole Floer theory. From this, we are naturally led to Lagrangian boundary conditions as we now explain. Define the following Chern-Simons-Dirac one-form $\mu = \mu_{CSD}$ on $T^*\mathfrak{C}(Y)$:

$$\mu(b,\psi) = ((b,\psi), SW_3(B,\Psi))_{L^2(Y)}, \qquad (b,\psi) \in T_{(B,\Psi)}\mathfrak{C}(Y) = \Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S}). \quad (9.17)$$

Here, the above L^2 inner product on $\Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S})$ is the one induced from the Riemannian inner product on Y and the real part of the Hermitian inner product on $\Gamma(\mathcal{S})$. If we take the differential of μ, we obtain

$$d_{(B,\Psi)}\mu((a,\phi),(b,\psi)) = ((a,\phi), \mathcal{H}_{(B,\Psi)}(b,\psi))_{L^2(Y)} - (\mathcal{H}_{(B,\Psi)}(a,\phi),(b,\psi))_{L^2(Y)} \qquad (9.18)$$

where $\mathcal{H}_{(B,\Psi)} : \Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \to \Omega^1(Y;i\mathbb{R}) \oplus \Gamma(\mathcal{S})$ is the "Hessian" operator obtained by differentiating the map SW_3 (of course, $\mathcal{H}_{(B,\Psi)}$ is only a true Hessian when Y is closed, because only then is SW_3 the gradient of a functional). Recall from Part I that the Hessian is given by

$$\mathcal{H}_{(B,\Psi)} = \begin{pmatrix} *_Y d & 2i\mathrm{Im}\,\rho^{-1}(\cdot\Phi^*)_0 \\ \rho(\cdot)\Phi & D_B \end{pmatrix}, \qquad (9.19)$$

a first order formally self-adjoint operator. Thus, (9.18) automatically vanishes on a closed manifold. However, integration by parts shows that when $\partial Y = \Sigma$ is nonempty, (9.18) defines a skew-symmetric pairing on the boundary. A simple computation shows that this pairing is the symplectic form ω in (9.1).

Define the (tangential) restriction map

$$\begin{aligned} r_\Sigma : \mathfrak{C}(Y) &\to \mathfrak{C}(\Sigma) \\ (B,\Psi) &\mapsto (B|_\Sigma, \Psi|_\Sigma) \end{aligned} \qquad (9.20)$$

The above discussion shows that if we pick a submanifold $\mathfrak{X} \subset \mathfrak{C}(Y)$ such that for every $(B,\Psi) \in \mathfrak{X}$, the space $r_\Sigma(T_{(B,\Psi)}\mathfrak{X})$ is an isotropic subspace of $T_{r_\Sigma(B,\Psi)}\mathfrak{C}(\Sigma)$, then $d\mu|_{\mathfrak{X}}$ vanishes. In particular, pick a Lagrangian submanifold $\mathfrak{L} \subset \mathfrak{C}(\Sigma)$ and define the space

$$\mathfrak{C}(Y,\mathfrak{L}) = \{(B,\Psi) \in \mathfrak{C}(Y) : r_\Sigma(B,\Psi) \in \mathfrak{L}\} \qquad (9.21)$$

consisting of those configurations on Y whose restriction to Σ lies in $\mathfrak{L}$. Then by the above considerations, μ is a closed 1-form when restricted to $\mathfrak{C}(Y,\mathfrak{L})$.

It is now possible to consider the Morse-Novikov theory for μ on $\mathfrak{C}(Y,\mathfrak{L})$. The resulting Floer equations, i.e., the formal downward flow of μ viewed as a vector field, are the

[4]We ignore the singular points of the quotient configuration space in this informal discussion.

equations (9.15) supplemented with the boundary condition

$$r_\Sigma(B(t), \Psi(t)) \in \mathfrak{L}, \qquad t \in \mathbb{R}. \tag{9.22}$$

Here, we choose a Lagrangian submanifold (rather than an isotropic one) because Lagrangian boundary conditions are precisely the ones that give rise to self-adjoint boundary conditions[5]. In other words, by supplying Lagrangian boundary conditions, the linearization of the system of equations (9.15) and (9.22) yields a time-dependent family of self-adjoint operators, from which it is then possible to compute the spectral flow of such a family (provided the requisite decay properties hold at infinity). This spectral flow makes it possible to assign a relative grading for the chain complex generated by the critical points in a Floer theory.

Summary of Analytic Difficulties

Let us shed some insight on the formidable analytic difficulties that the equations (9.7) pose, and in particular, let us compare these equations with the corresponding ASD equations studied in [54]. In both these situations, what we essentially have is an elliptic semilinear partial differential equation, with nonlinear, nonlocal boundary conditions. By elliptic, we mean that in a suitable gauge, the principal symbol of the equations on $\mathbb{R}\times Y$ are elliptic. By nonlocal, we that the Lagrangian boundary condition is not given by a set of differential equations on Σ. More precisely, a tangent space to our Lagrangian $\mathcal{L}$ is given not by the kernel of a differential operator but of a pseudodifferential operator (at least approximately, in the sense described in Part I). However, let us note that what is truly nonstandard about both these boundary problems is that the nonlocal boundary conditions are imposed "slice-wise", that is, they are specified pointwise in the time variable $t \in \mathbb{R}$. This implies that the linearization of the boundary condition (9.22) is determined (again, approximately) by the range of a product-type pseudodifferential operator, or more precisely, a time-dependent pseudodifferential operator on Σ, which is therefore, not pseudodifferential as an operator on $\mathbb{R} \times \Sigma$. We therefore have neither a local nor a nonlocal (pseudodifferential) boundary condition in the usual sense.

However, let us point out that in many ways, the ASD situation is "almost local" whereas in our situation, this is not at all the case. In both the ASD and Seiberg-Witten case, the action of the gauge group gives the "local part" of the Lagrangian (a gauge orbit and its tangent space are defined by differential equations) and dividing by the gauge group gives the remaining "nonlocal part" of the Lagrangian. However, in the ASD case, the Lagrangians modulo gauge are finite dimensional, whereas in the Seiberg-Witten case, they are infinite dimensional. Indeed, in the ASD case, the Lagrangians must lie in the space of flat connections, the zero set of the moment map associated to gauge group action on the space of $SU(2)$ connections on Σ. Hence, modulo gauge, these Lagrangians descend to Lagrangian submanifolds of the (singular) finite dimensional symplectically reduced space, the representation variety of Σ. On the other hand, because of the presence of spinors in

[5]Here in this informal discussion, we are being very loose with the precise functional analytic details, since self-adjointness requires that we find Lagrangians in the correct Hilbert space of boundary value data. See Sections 11 and 22 for a more rigorous discussion of the relationship between Lagrangians and self-adjointness.

the Seiberg-Witten case, the symplectic reduction of the moment map associated to the gauge group action on $\mathfrak{C}(\Sigma)$ is infinite dimensional. Hence, the Lagrangians one must consider descend to an infinite dimensional Lagrangian submanifold of this reduced space. For monopole Lagrangians in particular, the nonlocal part of Lagrangian is of a pseudodifferential nature (see Part I). Thus, our work here requires pseudodifferential analysis whereas the ASD case does not.

Moreover, it turns out that we are led to introduce some nonstandard function spaces because of the slicewise nature of the Lagrangian boundary condition. Such a boundary condition places time ($t \in \mathbb{R}$) and space (the manifold Y) on a different footing, and consequently, the estimates we perform on $\mathbb{R} \times Y$ will, in particular, measure regularity in time and space differently. Function spaces that distinguish among different directions are known as *anisotropic* function spaces (in contrast to the usual isotropic function spaces that measure the regularity of a function equally in every direction.) Specifically, we are required to work with both (isotropic) Besov spaces and also *anisotropic Besov spaces.* See Section 2 for a definition of these spaces.

Of course, our Lagrangians are not linear objects, and thus we will have to do a fair amount of nonlinear analysis in conjunction with the pseudodifferential nature of the our Lagrangians in the setting of anisotropic Besov spaces. This is in contrast to the ASD situation, where since the Lagrangians are finite dimensional modulo gauge, and all norms on a finite dimensional space are equivalent, there are no functional analytic difficulties posed by the nonlinearities of the Lagrangian. That is, in the ASD case, the nonlinearity of the Lagrangian only becomes a central issue in the bubbling analysis of [55], whose importance we described after Theorem B, and not the elliptic regularity analysis.

In fact, it is the necessity of such future bubbling analysis for the Seiberg-Witten case that requires that we work with the $H^{1,p}$ spaces in Theorems A and B. In [55], the bubbling analysis in the instanton case requires that the instanton analogs of Theorems A and B are proven for $H^{1,p}$ for $p > 2$, and not say, for $H^{k,2}$ with k large. Indeed, to exclude bubbling in the situations relevant for defining Floer homology, [55] applies mean value inequalities to deduce that the energy density of instantons remain bounded, and (near the boundary) bounds on the energy density give $H^{1,p}$ control of an instanton modulo gauge and not $H^{k,2}$ bounds for any $k > 1$. Thus, it is expected that future analysis of the bubbling phenomenon for the Seiberg-Witten equations will depend on $H^{1,p}$ analysis as well. Moreover, we should remark that proving our main theorems for $H^{k,2}$ regular solutions for large k does not simplify the analysis in any fundamental or conceptual way, since all the main technical steps, which we outline below, will still need to be performed.

Outline: Part III is organized as follows. In Section 10, we define the anisotropic function spaces we will be using. We then apply these function spaces to study the space of paths through a monopole Lagrangian $\mathfrak{L}$. This is necessary in light of the correspondence (9.12), which relates configurations on $\mathbb{R} \times Y$ to paths through the configuration space on Y. The boundary condition (9.22) specifies that on the boundary, such a path is a path through $\mathfrak{L}$. Here, we make use of the analysis developed in Part I in order to show that the space of paths through monopole Lagrangians is again a manifold in (anisotropic) Besov space topologies. The main theorems of this section, Theorems 10.9 and 10.9 are where we do our main nonlinear analysis on anisotropic Besov spaces. In Section 11, we show that the linearization of (9.7) is Fredholm in the periodic setting and in the appropriate function space topologies (including anisotropic ones). (That the linearization of (9.7) makes sense follows from the results of Section 2.) Here, a key step is to establish a resolvent estimate on anisotropic function spaces, which we need for the proof of Theorem 11.7. This resolvent estimate is established in Corollary 15.34, and its proof is the reason we need the parameter-dependent pseudodifferential calculus set up in Part IV. This resolvent estimate also relies crucially on aspects pertaining to self-adjointness, which in the end, amounts to the requirement that our boundary condition in (9.7) be given by a Lagrangian. In Section 12, we apply the tools from the previous sections to prove our main theorems, whose proof we summarize as follows. First, we work locally in time, which means we replace the time interval $\mathbb{R}$ in Theorem A with S^1 in Theorem 12.1. After this, the first step is to place the equations in a suitable gauge such that the linear part of the resulting equations is elliptic and falls into the framework of Section 11. From this, the second step is to gain regularity in the Σ directions for the gauge-fixed (A, Φ) in a neighborhood of the boundary using the anisotropic estimates of Sections 10 and 11. The third step is to gain regularity in the time direction and normal direction to Σ using the theory of Banach space valued Cauchy-Riemann equations due to Wehrheim [52]. Once we have gained some regularity in all directions, then in our final step, we bootstrap to gain regularity to any desired order, which proves Theorem 12.1. We then deduce Theorems A and B in Section 4 from Theorem 12.1. (Note if one starts with $H^{k,2}$ regularity in Theorem A for k large, one never has to work with $p \neq 2$ spaces and so on a first reading, one may assume that this is the case for simplicity. One still has to bootstrap in the above anisotropic fashion, however.)

Let us finally remark that the analysis we do is of a very general character and is likely to be applicable to other elliptic, semilinear boundary value problems whose linear part is a Dirac type operator and whose slicewise boundary condition (9.22) is given by a Lagrangian submanifold $\mathfrak{L}$ which satisfies formally similar properties to those obeyed by the monopole Lagrangians. Indeed, after one inspects the proof of Theorem A, one sees that the essential analysis we do has very little to do with the fact that we are dealing with the Seiberg-Witten equations per se. In the general situation, for operators of Dirac type on $\mathbb{R} \times Y$, then near the boundary, the part of the operator that differentiates in the $\mathbb{R}$ and normal directions becomes a Cauchy-Riemann operator. For this operator, we can apply the methods of Section 16 to gain regularity in the $\mathbb{R}$ and normal directions near the boundary so long as we have gained smoothness in the remaining Σ directions. Such regularity may be obtained by the anisotropic linear theory of Section 10 (where the anisotropy is in the Σ directions), so long as the tangent spaces to the Lagrangian $\mathfrak{L}$ fall within the framework of Section 10 and the nonlinear components of its chart maps for $\mathfrak{L}$ smooth in the Σ directions (i.e. the space of paths through $\mathfrak{L}$ satisfies properties similar to those in Theorems 10.9 and 10.10). It is not unreasonable to expect that any naturally occurring Lagrangians should satisfy such properties. Indeed, if they are locally determined by the zero locus of an analytic function involving multiplication and pseudodifferential–like operators (as monopole Lagrangians are), then as done in Section 10, an analysis of the nonlinear compositions of multiplication and pseudodifferential–like operators which enter into the analysis of their chart maps should allow us to recover suitable smoothing properties. Nevertheless, the analysis we had to do in Section 10 is quite difficult, and it would have been unwise to obscure the technical exposition of Section 10 by writing it for abstract Lagrangians, even though after the fact, one is led to believe that the results there should hold more generally.

Of course, constructing a monopole Floer theory with Lagrangian boundary conditions will have to depend on the precise nature of the Seiberg-Witten equations and the chosen Lagrangian. In this regard, the Lagrangians we have chosen, namely the monopole Lagrangians, are the most natural ones to consider.

10 Spaces of Paths

A smooth path $(B(t),\Phi(t)) \in \mathrm{Maps}(\mathbb{R},\mathfrak{C}(Y))$ satisfying the boundary condition (9.22) implies that we get a path

$$r_\Sigma(B(t),\Phi(t)) \in \mathrm{Maps}(\mathbb{R},\mathfrak{L})$$

through the smooth Lagrangian $\mathfrak{L}$, which we assume to be a monopole Lagrangian $\mathfrak{L} = \mathcal{L}(Y',\mathfrak{s}')$. Our task in this section is to show that the space of paths through $\mathfrak{L}$ in (anisotropic) Besov space topologies forms a Banach manifold obeying analytic properties suitable for proving Theorems A and B. The main theorems of this section are Theorems 10.9 and 10.10. From Part I, given a monopole Lagrangian $\mathcal{L}$, then $\mathcal{L}^{s-1/p,p}$, the $B^{s-1/p,p}(\Sigma)$ clsoure of $\mathcal{L}$, is a smooth Banach submanifold of $\mathfrak{C}^{s-1/p,p}(\Sigma)$, the $B^{s-1/p,p}(\Sigma)$ closure of $\mathfrak{C}(\Sigma)$ for $s > \max(3/p, 1/2+1/p)$. Furthermore, the local chart maps of $\mathcal{L}^{s-1/p,p}$ are described by Theorem 4.15. In this theorem, the nonlinear part of a particular chart map at a configuration $u \in \mathcal{L}^{s-1/p,p}$, which we denoted by E^1_u, is smoothing. In other words, while every Banach manifold is locally a graph over its tangent space, our monopole Lagrangian possesses charts that are graphs of maps which increase regularity. However, if we now consider the space of paths through our Lagrangians, then since the corresponding chart maps on the space of paths will be defined "slicewise" along the path (see Definition 10.4), the smoothing continues to occur but only in the space variables Σ and not in the time variable. This naturally leads us to consider spaces which have extra smoothness in some directions and hence anisotropic spaces. Because our monopole spaces are modeled on Besov spaces, we thus end up with anisotropic Besov spaces.

In this section, we consider the Lagrangians $\mathfrak{L} = \mathcal{L}(Y,\mathfrak{s})$, which we abbreviate as $\mathcal{L}$, in generality, where Y is any 3-manifold and $\mathfrak{s}$ is any spin^c structure on Y such that (4.1) holds. When we return to the Seiberg-Witten equations on $\mathbb{R}\times Y$ in the next section, we will make use of monopole Lagrangians $\mathcal{L}(Y',\mathfrak{s}')$, with Y' and $\mathfrak{s}'$ satisfying Definition 9.2.

Let us recall the basic notation and setup of the results in Part I so that we can analyze how $\mathcal{L}$ is constructed. With $\mathfrak{s}$ fixed, we have the configuration spaces

$$\mathfrak{C}(Y) = \mathcal{A}(Y) \times \Gamma(\mathcal{S}) \tag{10.1}$$

$$\mathfrak{C}(\Sigma) = \mathcal{A}(\Sigma) \times \Gamma(\mathcal{S}_\Sigma), \tag{10.2}$$

of connections and spinors on Y and Σ, where $\mathcal{S}$ is the spinor bundle on Y associated to $\mathfrak{s}$

and $\mathcal{S}_\Sigma$ is the restriction of $\mathcal{S}$ to Σ. Both of these spaces are affine spaces modeled on

$$\mathcal{T} = \mathcal{T}_Y = \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \tag{10.3}$$
$$\mathcal{T}_\Sigma = \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma), \tag{10.4}$$

respectively, and the tangential restriction map (2.5) on configuration spaces induces one on the tangent spaces:

$$\begin{aligned} r_\Sigma &: \mathcal{T} \to \mathcal{T}_\Sigma \\ &(b, \psi) \mapsto (b|_\Sigma, \psi|_\Sigma). \end{aligned} \tag{10.5}$$

The space of monopoles $\mathfrak{M} = \mathfrak{M}(Y, \mathfrak{s})$ is the zero set of the Seiberg-Witten map SW_3 given by (9.16). Fixing a smooth reference connection B_{ref}, we have the space

$$\mathcal{M} = \{(B, \Psi) \in \mathfrak{M} : d^*(B - B_{\mathrm{ref}}) = 0\} \tag{10.6}$$

of monopoles in Coulomb gauge with respect to B_{ref}. The space $\mathcal{L}$ is the space of boundary values of $\mathfrak{M}$, which is equal to the space of boundary values of $\mathcal{M}$, i.e.,

$$\mathcal{L} = r_\Sigma(\mathfrak{M}) = r_\Sigma(\mathcal{M}), \tag{10.7}$$

where r_Σ is the tangential restriction map (2.5).

All these definitions extend to the appropriate Besov completions, as was done in Part I. Thus, we have the configuration spaces $\mathfrak{C}^{s,p}(Y)$ and $\mathfrak{C}^{s,p}(\Sigma)$, the $B^{s,p}(Y)$ and $B^{s,p}(\Sigma)$ closures of $\mathfrak{C}(Y)$ and $\mathfrak{C}(\Sigma)$, respectively. We also have the following Besov monopole spaces

$$\mathfrak{M}^{s,p} = \{(B, \Psi) \in \mathfrak{C}^{s,p}(Y) : SW_3(B, \Psi) = 0\}, \tag{10.8}$$
$$\mathcal{M}^{s,p} = \{(B, \Psi) \in \mathfrak{C}^{s,p}(Y) : SW_3(B, \Psi) = 0, d^*(B - B_{\mathrm{ref}}) = 0\}, \tag{10.9}$$
$$\mathcal{L}^{s-1/p,p} = r_\Sigma(\mathcal{M}^{s-1/p,p}). \tag{10.10}$$

From Part I, for $p \geq 2$ and $s > \max(3/p, 1/2)$, the spaces $\mathfrak{M}^{s,p}$ and $\mathcal{M}^{s,p}$ are Banach submanifolds of $\mathfrak{C}^{s,p}(Y)$. If in addition, $s > \max(3/p, 1/2 + 1/p)$, then $\mathcal{L}^{s-1/p,p}$ is a Banach submanifold of $\mathfrak{C}^{s-1/p,p}(\Sigma)$ and

$$r_\Sigma : \mathcal{M}^{s,p} \to \mathcal{L}^{s-1/p,p} \tag{10.11}$$

is a covering.

When $\mathcal{L}^{s-1/p,p}$ is a Banach manifold, the space $C^0(I, \mathcal{L}^{s-1/p,p})$ of continuous paths from an interval I into $\mathcal{L}^{s-1/p,p}$ is naturally a Banach manifold. However, it is far from obvious that the space of paths through $\mathcal{L}^{s-1/p,p}$ in anisotropic Besov topologies is a Banach manifold. We now define these anisotropic Besov spaces precisely.

10.1 Anisotropic Function Space Setup

On Euclidean space, the usual Besov spaces $B^{s,p}(\mathbb{R}^n)$ are well-defined, for any $s \in \mathbb{R}$ and $1 < p < \infty$, see Part IV. Recall that for $p = 2$, the spaces $B^{s,2}(\mathbb{R}^n)$ coincide with the L^2 Sobolev spaces $H^{s,2}(\mathbb{R}^n)$. Suppose we have a splitting $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then for any

$s_1 \in \mathbb{R}$ and $s_2 \geq 0$, we define the anisotropic Besov space $B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as follows. Let $x = (x_{(1)}, x_{(2)})$ be the coordinates on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and let $\xi = (\xi_{(1)}, \xi_{(2)})$ be the corresponding Fourier variables. Let $\mathcal{F}_2$ denote Fourier transform on $\mathbb{R}^{n_2}$ and define the operator

$$J_{(2)}f = \mathcal{F}_2 \left(1 + \left|\xi_{(2)}\right|^2\right)^{1/2} \mathcal{F}_2^{-1} f$$

for any smooth compactly supported f. Thus, very roughly speaking, for any $s \geq 0$, $J_{(2)}^s$ takes s derivatives in all the $\mathbb{R}^{n_2}$ directions.

Definition 10.1 For $s_1 \in \mathbb{R}$ and $s_2 \geq 0$, the anisotropic Besov space $B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined to be closure of smooth compactly supported functions on $\mathbb{R}^n$ with respect to the norm

$$\|f\|_{B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} := \|J_{(2)}^{s_2} f\|_{B^{s_1,p}(\mathbb{R}^n)}. \tag{10.12}$$

Equivalently (see [36]), the space $B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is the space of all functions lying in the classical (isotropic) Besov space $B^{s_1,p}(\mathbb{R}^n)$ for which $D_{s_2} f \in B^{s_1,p}(\mathbb{R}^n)$, where D_{s_2} is any smooth elliptic operator on $\mathbb{R}^{n_2}$ of order s_2. That is, we have the equivalence of norms

$$\|f\|_{B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \sim \|f\|_{B^{s_1,p}(\mathbb{R}^n)} + \|D_{s_2} f\|_{B^{s_1,p}(\mathbb{R}^n)}. \tag{10.13}$$

When $s_2 = 0$, then $B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is just the usual Besov space $B^{s_1,p}(\mathbb{R}^n)$.

The fundamental properties of these spaces are worked out in Part IV. As described there, in the usual way we may then define anisotropic function spaces on products of open sets in Euclidean space and hence on products of manifolds (with boundary). Most of the analysis in this section will occur on the anisotropic spaces $B^{(s_1,s_2),p}(I \times Y)$ and $B^{(s_1,s_2),p}(I \times \Sigma)$ where I is an interval. Indeed, as we have mentioned, we will be considering maps that smooth in the space variables Y and Σ.

We apply the anisotropic function space setup as follows. Let I be a time interval, bounded or infinite. We have the smooth configuration space

$$\begin{aligned} \mathfrak{C}(I \times Y) &= \mathcal{A}(I \times Y) \times \Gamma(I \times \mathcal{S}) \\ &= \mathrm{Maps}(I, \mathcal{A}(Y) \times \Gamma(\mathcal{S}) \times \Omega^0(Y; i\mathbb{R})) \\ &= \mathrm{Maps}(I, \mathfrak{C}(Y) \times \Omega^0(Y; i\mathbb{R})) \end{aligned} \tag{10.14}$$

via the correspondence (9.10). Here, given any space X, we write $\mathrm{Maps}(I, X)$ to denote the space of smooth maps from I into X. Likewise, we write $C^0(I, X)$ to denote the space of continuous maps from I into X. We refer to such maps as paths. Next, replacing Y with Σ in the above, we have

$$\begin{aligned} \mathfrak{C}(I \times \Sigma) &= \mathcal{A}(I \times \Sigma) \times \Gamma(I \times \mathcal{S}_\Sigma) \\ &= \mathrm{Maps}(I, \mathfrak{C}(\Sigma) \times \Omega^0(\Sigma; i\mathbb{R})). \end{aligned} \tag{10.15}$$

Since we have the anisotropic function spaces $B^{(s_1,s_2),p}(I \times Y)$ and $B^{(s_1,s_2),p}(I \times \Sigma)$, we get induced topologies on the configuration spaces, their subspaces, and their corresponding tangent spaces.

Notation 10.2 Let $\mathfrak{X}$ be a space of configurations over the space $X_1 \times X_2$, where X_1 and X_2 are as in Definition 13.21. Then $B^{(s_1,s_2),p}\mathfrak{X}$ denotes the closure of $\mathfrak{X}$ in the $B^{(s_1,s_2),p}(X_1 \times X_2)$ topology. Likewise, if $\mathfrak{X}$ is a space of configurations over a manifold X, we let $B^{s,p}\mathfrak{X}$ denote the closure of $\mathfrak{X}$ in the $B^{s,p}(X)$ topology. If E is a vector bundle over $X = X_1 \times X_2$, we write $B^{(s_1,s_2),p}(E)$ as shorthand for $B^{(s_1,s_2),p}\Gamma(E)$. Similar definitions apply for prefixes given by other topologies, e.g., L^p and C^0. We also write $E_{\partial X} = E|_{\partial X}$ for the bundle E restricted to ∂X. Finally, we will sometimes refer to just the topology of a configuration, e.g., we will say an element of $B^{s,p}\mathfrak{X}$ belongs to $B^{s,p}$.

Thus, letting $M = Y$ or Σ, we can consider the anisotropic configuration spaces

$$\mathfrak{C}^{(s_1,s_2),p}(I \times M) := B^{(s_1,s_2),p}\mathfrak{C}(I \times M) \tag{10.16}$$

and we can consider their anisotropic tangent spaces

$$\begin{aligned} T_\bullet\mathfrak{C}^{(s_1,s_2),p}(I \times M) &= B^{(s_1,s_2),p}(\Omega^1(I \times M; i\mathbb{R}) \oplus \Gamma(I \times \mathcal{S}_M)), &(10.17)\\ &= B^{(s_1,s_2),p}\left(\mathrm{Maps}(I, \mathcal{T}_M) \times \mathrm{Maps}(I, \Omega^0(M; i\mathbb{R}))\right), &(10.18)\end{aligned}$$

where $\bullet$ is any basepoint. Here, we used (10.14), (10.15), (10.3), and (10.4). These anisotropic spaces induce corresponding topologies on their subspaces, in particular, those subspaces given by the space of paths through $\mathfrak{C}(M)$ and $\mathcal{T}_M$, respectively. Thus, the spaces $B^{(s_1,s_2),p}\mathrm{Maps}(I, \mathfrak{C}(M))$ and $B^{(s_1,s_2),p}(I, \mathcal{T}_M)$ are topologized as subspaces of (10.16) and (10.18), respectively. Moreover, all these spaces are the completions of spaces of smooth configurations in the $B^{(s_1,s_2),p}(I \times M)$ topology, as is consistent with Notation 10.2.

The above definitions work out nicely because the spaces we are topologizing are affine spaces. Suppose we now wish to topologize spaces that are not linear, namely, the space of paths through $\mathcal{M}$ and $\mathcal{L}$. For this, we can describe more general path spaces in anisotropic topologies as follows. By the trace theorem, Theorem 13.22, we have a trace map

$$r_t : B^{(s_1,s_2),p}(\mathbb{R} \times M) \to B^{s_1+s_2-1/p,p}(\{t\} \times M), \qquad t \in \mathbb{R} \tag{10.19}$$

for all $s_1 > 1/p$. Moreover, this trace map is continuous in t, in other words, we have the inclusion

$$B^{(s_1,s_2),p}(\mathbb{R} \times M) \hookrightarrow C^0(\mathbb{R}; B^{s_1+s_2-1/p,p}(M)). \tag{10.20}$$

Thus, we have the following well-defined space of paths in anisotropic topologies:

Definition 10.3 Let $s_1 > 1/p$, $s_2 \geq 0$, and $p \geq 2$. Then we can define the following spaces

$$\mathrm{Maps}^{(s_1,s_2),p}(I, \mathfrak{C}(M)) = B^{(s_1,s_2),p}\mathrm{Maps}(I, \mathfrak{C}(M)) \tag{10.21}$$

$$\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{T}_M) = B^{(s_1,s_2),p}\mathrm{Maps}(I, \mathcal{T}_M) \tag{10.22}$$

$$\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{M}) = \{\gamma \in \mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{T}_Y) : \gamma(t) \in \mathcal{M}^{s_1+s_2-1/p,p}, \text{ for all } t \in I\} \tag{10.23}$$

$$\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{L}) = \{\gamma \in \mathrm{Maps}^{(s_1,s_2),p}(I, \mathfrak{C}(\Sigma)) : \gamma(t) \in \mathcal{L}^{s_1+s_2-1/p,p}, \text{ for all } t \in I\}, \tag{10.24}$$

If $s_2 = 0$, we write $\mathrm{Maps}^{s_1,p}$ instead of $\mathrm{Maps}^{(s_1,0),p}$. Similar definitions apply if we extend $\mathfrak{C}(M)$ and $\mathcal{T}_M$ in the above by vector bundles over M, in particular, $\Omega^0(M; i\mathbb{R})$. Finally, if X is a subspace of $\mathcal{T}_M^{s,p}$ for some s, define

$$\mathrm{Maps}^{(s_1,s_2),p}(I, X) = B^{(s_1,s_2),p}\{z \in \mathrm{Maps}(I, \mathcal{T}_M) : z(t) \in X, \text{ for all } t \in I\}, \tag{10.25}$$

the $B^{(s_1,s_2),p}(I, M)$ closure of the space of paths in $\mathrm{Maps}(I, \mathcal{T}_M)$ which take values in X. Note (10.25) generalizes definition (10.22).

The first two definitions above are just a change of notation since we have already defined the right-hand side. However, for the next two definitions, there is a subtle point in how we defined these spaces as compared to the previous ones. Observe that for (10.21) and (10.22), the space $\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{T}_M)$, say, is *defined* to be the closure of a space of smooth paths, namely, the closure of $\mathrm{Maps}(I, \mathcal{T}_M)$ in the $B^{(s_1,s_2),p}(I \times M)$ topology. On the other hand, in (10.24) say, we begin with a path γ in the $B^{(s_1,s_2),p}(I \times \Sigma)$ topology, and we impose a restriction on its trace, namely that it always lie in $\mathcal{L}^{s_1+s_2-1/p,p}$. The resulting space is a priori a larger space than the closure of the space of smooth paths through the space $\mathcal{L}$, i.e., we have

$$B^{(s_1,s_2),p}\mathrm{Maps}(I, \mathcal{L}) \subseteq \mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{L}). \tag{10.26}$$

Indeed, to prove the reverse inequality, one would have to approximate an arbitrary path γ in the space $\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{L})$ by a smooth path (both in space *and* time), which because of the nonlinearity of $\mathcal{L}$, is not obvious how to do. Indeed, even though we showed in Part I that the smooth monopole space $\mathcal{L}$ is dense in $\mathcal{L}^{s_1+s_2-1/p}$, this says nothing about the temporal regularity of an approximating sequence to the space of paths. However, by Theorems 10.9 and 10.10, it does turn out to be the case that we can approximate the space of paths by smooth paths in the appropriate range of s_1, s_2, and p, so that we have equality in (10.26).

We now have all the appropriate definitions and notation in place for our anisotropic path spaces. Our main task in the rest of this section is to prove the analogous results in Part I for the space of paths through $\mathcal{L}$, i.e., we want to prove that the space $\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{L})$ is a Banach manifold for a large range of parameters and it has chart maps with smoothing properties as described in the discussion at the beginning of this section. These chart maps are defined from the charts for $\mathcal{L}$ in a "slicewise fashion" along a path in $\mathrm{Maps}^{(s_1,s_2),p}(I, \mathcal{L})$. We thus need to understand the analytic properties of such slicewise operators.

10.2 Slicewise Operators on Paths

Unless stated otherwise, from now on, we always assume $p \geq 2$. In the following, we will be estimating the operators studied in Part I acting slicewise on the space of paths from an interval I into some target configuration space. More precisely, we have the following definition.

Definition 10.4 Let $O(t)$ be a family of operators acting on a configuration space $\mathfrak{X}$, $t \in I$. We write $\widehat{O(t)}$ to denote the slicewise operator associated to the family $O(t)$, that is, $\widehat{O(t)}$

applied to a path $\gamma : I \to \mathfrak{X}$ yields the path

$$\widehat{O(t)}\gamma = \Big(t \mapsto O(t)\gamma(t)\Big)_{t\in I}. \tag{10.27}$$

Notation 10.5 Note that given a path of configurations γ, if we write $\gamma = \gamma(t)$, it is ambiguous whether we mean the whole path as a function of t or just the single configuration at time t. The above hat notation mitigates this ambiguity. Moreover, we will mainly be considering the case when $O(t) \equiv O$ is time-independent. In this case, the $\widehat{\ }$ notation therefore just serves as a notational reminder, although in certain cases, such as when O is a differential operator, we will sometimes just write O to denote O acting slicewise, which is standard notational practice in this case.

We now proceed to estimate the linear operators of interest to us when they act slicewise. Recall that we defined the notion of a *local straightening map* in Definition 20.3 in an abstract framework, and showed how local straightening maps yield natural local charts for Banach submanifolds of a Banach space. In Lemma 4.14, we defined a local straightening map $F_{\Sigma,(B_0,\Psi_0)}$ for $\mathcal{L}^{s-1/p,p}$ at a configuration $r_\Sigma(B_0,\Psi_0) \in \mathcal{L}^{s-1/p,p}$, where $(B_0,\Psi_0) \in \mathcal{M}^{s,p}$. In this way, we used the local straightening map $F_{\Sigma,(B_0,\Psi_0)}$ to obtain chart maps for the Banach submanifold $\mathcal{L}^{s-1/p,p} \subset \mathfrak{C}^{s-1/p,p}(\Sigma)$ in Theorem 4.15. We also deduced some important properties of the resulting chart maps, $E_{r_\Sigma(B_0,\Psi_0)}$, namely that they are defined on large domains (i.e. $B^{s',p}(\Sigma)$ open balls) and that the nonlinear portion of the chart map $E^1_{r_\Sigma(B_0,\Psi_0)}$ smooths by a derivative. When studying the space of paths $\mathrm{Maps}^{s,p}(I,\mathcal{L})$, our first goal is to show that the slicewise map $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ gives a local straightening map within a neighborhood of a constant path identically equal to $r_\Sigma(B_0,\Psi_0) \in \mathcal{L}^{s,p}$. From this, the local straightening map yields for us a chart map for $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ at a constant path. Later, we will see how to "glue together" these chart maps for constant paths on small time intervals to obtain a chart map at an arbitrary path in $\mathrm{Maps}^{s,p}(I,\mathcal{L})$.

One of the operators that arises in the definition of $F_{\Sigma,(B_0,\Psi_0)}$, as seen in (4.48) and (4.29), is the operator $Q_{(B_0,\Psi_0)}$, defined in (4.21). Let us review this operator. The operator $Q_{(B_0,\Psi_0)}$ is constructed out of the Hessian operator $\mathcal{H}_{(B_0,\Psi_0)}$ (more precisely, its inverse on suitable domains), the projection $\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}$, and a pointwise quadratic multiplication map q. Let us briefly review these maps. Recall that for any configuration $(B,\Psi) \in \mathfrak{C}^{s,p}(Y)$, the operator

$$\mathcal{H}_{(B,\Psi)} : \mathcal{T}^{s,p} \to \mathcal{T}^{s-1,p} \tag{10.28}$$

is given by

$$\mathcal{H}_{(B,\Psi)} = \begin{pmatrix} *_Y d & 2i\mathrm{Im}\,\rho^{-1}(\cdot\Phi^*)_0 \\ \rho(\cdot)\Phi & D_B \end{pmatrix}. \tag{10.29}$$

It is a first order formally self-adjoint operator and it has kernel $T_{(B,\Psi)}\mathfrak{M}^{s,p}$ whenever $(B,\Psi) \in \mathfrak{M}^{s,p}$. By Lemma 4.5, given any $(B_0,\Psi_0) \in \mathfrak{M}^{s,p}$, we can choose a subspace $X^{s,p}_{(B_0,\Psi_0)} \subset \mathcal{T}^{s,p}$ complementary to $T_{(B,\Psi)}\mathfrak{M}^{s,p}$. For such $X^{s,p}_{(B_0,\Psi_0)}$, we have by Proposition

3.20 that

$$\mathcal{H}_{(B_0,\Psi_0)} : X^{s,p}_{(B_0,\Psi_0)} \to \mathcal{K}^{s-1,p}_{(B_0,\Psi_0)} \tag{10.30}$$

$$\mathcal{H}_{(B_0,\Psi_0)} : X^{s+1,p}_{(B_0,\Psi_0)} \to \mathcal{K}^{s,p}_{(B_0,\Psi_0)}, \qquad X^{s+1,p}_{(B_0,\Psi_0)} = X^{s,p}_{(B_0,\Psi_0)} \cap \mathcal{T}^{s+1,p} \tag{10.31}$$

are isomorphisms. Here, the subspace $\mathcal{K}^{s,p}_{(B_0,\Psi_0)} \subset \mathcal{T}^{s,p}$ is a complement to the tangent space of the gauge orbit at (B_0, Ψ_0) in $\mathcal{T}^{s,p}$, see Lemma 3.4. The map

$$\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : \mathcal{T}^{s,p} \to \mathcal{K}^{s,p}_{(B_0,\Psi_0)} \tag{10.32}$$

is a bounded projection onto this space, see (3.45). Finally, the map q arises from the quadratic multiplication that occurs in the map SW_3, see 1-(4.16). Thus, both q and $\widehat{\mathsf{q}}$ are pointwise multiplication operators, and their mapping properties are controlled by the function space multiplication theorem, Theorem 13.18.

Thus, when we consider the above operators slicewise, the main operators we need to understand are

$$(\widehat{\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}})^{-1}, \quad \Pi_{\widehat{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}}. \tag{10.33}$$

However, since the operators in (10.33) are time-independent, estimating them on (anisotropic) Besov spaces is not difficult. Indeed, we have the following general lemma:

Lemma 10.6 *Let $s_1 > 0$, and $s_2, s_2' \geq 0$, and let M be a compact manifold. Let $T : C^\infty(M) \to C^\infty(M)$ be a linear operator such it extends to a bounded operators*

$$T : H^{s_2,p}(M) \to H^{s_2+s_2',p}(M) \tag{10.34}$$

$$T : B^{s_1+s_2,p}(M) \to B^{s_1+s_2+s_2',p}(M). \tag{10.35}$$

Then the slicewise operator

$$\widehat{T} : B^{(s_1,s_2),p}(I \times M) \to B^{(s_1,s_2+s_2'),p}(M) \tag{10.36}$$

is bounded and the operator norm of (10.36) is bounded in terms of the operator norms of (10.34) and (10.35).

Proof The crucial property we need is the so-called "Fubini property" of Besov spaces (see [51]). Namely, for any $s > 0$, the Besov space $B^{s,p}(I \times M)$ can be written as the intersection

$$B^{s,p}(I \times M) = L^p(I, B^{s,p}(M)) \cap L^p(M, B^{s,p}(I)). \tag{10.37}$$

In other words, we can separate variables so that a function of regularity of order s on $I \times M$ is a function that has regularity of order s in I and M, separately, in the Besov sense. This Fubini property automatically implies one for anisotropic Besov spaces, and we have from (10.13) that

$$B^{(s_1,s_2),p}(I \times M) = L^p(I, B^{s_1+s_2,p}(M)) \cap H^{s_2,p}(M, B^{s_1,p}(I)). \tag{10.38}$$

Recall that $H^{s_2,p}(M)$ is the fractional Sobolev space of functions whose fractional derivatives up to order s_2 belong to $L^p(M)$.[6] In other words, given an elliptic differential operator D_{s_2} on M of order s_2, we can define the norm on $H^{s_2,p}(M)$ by

$$\|f\|_{H^{s_2,p}(M)} = \|f\|_{L^p(M)} + \|D_{s_2}f\|_{L^p(M)}.$$

Thus, the space $H^{s_2,p}(M, B^{s,p}(I))$ appearing in (10.38) is the space of functions f such that both f and $D_{s_2}f$ belong to $L^p(M, B^{s,p}(I))$.

We want to show that (10.36) is bounded. We proceed by using the decomposition (10.38). First, the boundedness of $T : B^{s_1+s_2,p}(M) \to B^{s_1+s_2+s_2',p}(M)$ implies the boundedness of

$$\widehat{T} : L^p(I, B^{s_1+s_2,p}(M)) \to L^p(I, B^{s_1+s_2+s_2',p}(M)). \tag{10.39}$$

It remains to show that the map

$$\widehat{T} : H^{s_2,p}(M, B^{s_1,p}(I)) \to H^{s_2+s_2',p}(M, B^{s_1,p}(I)) \tag{10.40}$$

is bounded. To do this, we first show that the space $H^{s_2,p}(M, B^{s,p}(I))$, as defined above, is also equal to the space $B^{s,p}(I, H^{s_2,p}(M))$ which is defined as follows.

Recall that while we defined Besov spaces $B^{s,p}$ in Part IV in terms of a Littlewood-Paley decomposition, there is an equivalent description in terms of finite difference operators. Namely, for $s > 0$, a norm on $B^{s,p}(\mathbb{R})$ is given by

$$\|f\|_{B^{s,p}(\mathbb{R})} = \left(\int_0^\infty \left(\int_{\mathbb{R}} |h^{-sm}(\tau_h - \mathrm{id})^m f(t)|^p dt\right) \frac{1}{h} dh\right)^{1/p},$$

where $m > s$ is any integer, and τ_h is the translation operator $(\tau_h f)(t) = f(t+h)$. This allows us to describe Banach space valued Besov spaces, namely

$$\|f\|_{B^{s,p}(\mathbb{R},X)} := \left(\int_0^\infty \left(\int_{\mathbb{R}} |h^{-sm}(\tau_h - \mathrm{id})^m f(t)|_X^p dt\right) \frac{1}{h} dh\right)^{1/p}. \tag{10.41}$$

For $X = H^{s_2,p}(M)$, we have $B^{s,p}(\mathbb{R}, H^{s_2,p}(M)) = H^{s_2,p}(M, B^{s_2,p}(\mathbb{R}))$ since the operators $h^{-sm}(\tau_h - \mathrm{id})^m$ and D_{s_2} commute. We now define $B^{s,p}(I, X)$ to be the restrictions of elements of $B^{s,p}(\mathbb{R}, X)$ to the domain I.

Thus, showing (10.40) is the same thing as showing

$$\widehat{T} : B^{s,p}(I, H^{s_2,p}(M)) \to B^{s,p}(I, H^{s_2+s_2',p}(M)). \tag{10.42}$$

To show (10.40), we proceed by interpolation. Namely, Besov spaces are interpolation spaces of Bessel potential spaces, i.e.,

$$B^{s,p}(\mathbb{R}) = (H^{t_0,p}(\mathbb{R}), H^{t_1,p}(\mathbb{R}))_{\theta,p}, \tag{10.43}$$

for any $t_0, t_1 \in \mathbb{R}$ and $0 < \theta < 1$ such that $(1-\theta)t_0 + \theta t_1 = s$. Here, $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor (see Part IV). Thus, since (10.43) holds, this means that if we have

[6] The space $H^{s_2,p}(M)$ is also known as a *Bessel potential* space. See Part IV for a precise definition in the general case.

bounded maps $T : H^{t_i,p} \to B^{t_i,p}$ for the endpoint spaces, $i = 0, 1$, then $T : B^{s,p} \to B^{s,p}$ is bounded and its operator norm can be bounded in terms of the operator norms of the endpoint operators.

Let t_0 and t_1 in the above be nonnegative integers. Then (10.43) also holds for Banach space valued Besov spaces[7]:

$$B^{s,p}(\mathbb{R}, X) = (H^{t_0,p}(X), H^{t_1,p}(X))_{\theta,p}, \tag{10.44}$$

Here, for k a nonnegative integer, the space $H^{k,p}(\mathbb{R}, X)$ is the Banach space of functions from $\mathbb{R}$ to X equipped the norm

$$\|f\|_{H^{k,p}(\mathbb{R},X)} = \left(\sum_{j=0}^{k} \int_{\mathbb{R}} \|\partial_t^k f\|_X^p\right)^{1/p}. \tag{10.45}$$

Thus, to establish that (10.42) is bounded, from interpolation, it suffices to establish that

$$\widehat{T} : H^{k,p}(I, H^{s_2,p}(M)) \to H^{k,p}(I, H^{s_2+s_2',p}(M)). \tag{10.46}$$

is bounded for all integer $k \geq 0$. For $k = 0$, this follows trivially from (10.34), and for $k \geq 1$, this also follows from (10.34) by commuting derivatives with T, since T is linear and time-independent (observe that if T were time-dependent but also smooth, this argument would follow too). Thus, this proves that (10.46) is bounded, which finishes the proof that (10.36) is bounded. □

The above lemma tells us that time-independent slicewise operators on $I \times M$ can be estimated in terms of their mapping properties on M. In particular, we can now easily estimate the slicewise operators in (10.33) on $I \times Y$ because we know how they act on Y from the analysis carried out in Part I. We have the following corollary:

Corollary 10.7 *Let $s > \max(3/p, 1/2)$ and $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$.*

(i) We have bounded maps

$$\left(\widehat{\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}}\right)^{-1} \Pi_{\widehat{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}} : \mathrm{Maps}^{s,p}(I, \mathcal{T}) \to \mathrm{Maps}^{(s,1),p}(I, \mathcal{T}) \tag{10.47}$$

(ii) Assume in addition that $(B_0, \Psi_0) \in \mathcal{M}^{s_1+s_2,2}$, where $s_1 \geq 3/2$ and $s_2 \geq 0$. Then we have

$$\left(\widehat{\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s_1+s_2+1,2}_{(B_0,\Psi_0)}}}\right)^{-1} \Pi_{\widehat{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}} : \mathrm{Maps}^{(s_1,s_2),2}(I, \widetilde{\mathcal{T}}) \to \mathrm{Maps}^{(s_1,s_2+1),2}(I, \widetilde{\mathcal{T}}) \tag{10.48}$$

[7]Here, we assume X is a UMD Banach space (see Part IV), which will always be the case for us.

Proof By the above lemma, to establish (10.47), it suffices to show that the maps

$$\left(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}\right)^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : L^p\mathcal{T} \to H^{1,p}\mathcal{T}$$
$$\left(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}\right)^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : \mathcal{T}^{s,p} \to \mathcal{T}^{s+1,p}$$

are bounded. (By the first line above, we of course mean that the operator in the second line extends to a bounded operator on the stated domain and ranges. Such notation will always be understood from now on.) The second line follows from (10.32) and (10.31). The first line follows from Lemma 4.7(ii). Likewise, for $(B_0,\Psi_0) \in \mathcal{M}^{s_1+s_2+1,2}$, (10.48) follows from

$$\left(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}\right)^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : L^2\mathcal{T} \to H^{1,2}\mathcal{T}$$
$$\left(\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}\right)^{-1}\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}} : \mathcal{T}^{s_1+s_2,2} \to \mathcal{T}^{s_1+s_2+1,2},$$

which also follows from 4.7(ii) and Proposition 3.20. □

The above estimates for slicewise operators will now allow us to pass from the local chart maps for $\mathcal{L}$ at a configuration (B_0,Ψ_0) to a local chart map for path $\gamma(t) = (B(t),\Psi(t))$ through $\mathcal{L}$. More precisely, we construct such charts in the Besov topology $B^{s,p}(I\times\Sigma)$ topology, and this shows that the space $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ is a Banach submanifold of $\mathrm{Maps}^{s,p}(I,\mathfrak{C}(\Sigma))$. Recall from earlier discussion that we construct such charts for $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ from *local straightening maps* (see Definition 20.3) for this space. Moreover, the nonlinear part of the chart maps for $\mathrm{Maps}^{s,p}(I,\mathcal{L})$, just like the nonlinear part of the the chart maps for $\mathcal{L}^{s-1/p,p}$ studied in Part I, will be smoothing in the Σ directions. This is because by Corollary 10.7, the slicewise inverse Hessian smooths in the Y directions, and when we restrict from $I\times Y$ to $I\times\Sigma$, the anisotropic trace results in Theorem 13.22 tell us that we preserve a portion of our gain in regularity in the Σ directions. We begin by proving the following lemma, which, in particular, gives us local straightening maps for constant paths in $\mathrm{Maps}^{s,p}(I,\mathcal{L})$. This is the slicewise analog of Lemma 4.14.

Lemma 10.8 *Let $s > \max(3/p, 1/2+1/p)$ and let $r_\Sigma(B_0,\Psi_0) \in \mathcal{L}^{s-1/p,p}$, where $(B_0,\Psi_0) \in \mathcal{M}^{s,p}$. Define the spaces*

$$\hat{X}_\Sigma = C^0(I,\mathcal{T}_\Sigma^{s-1/p,p}), \qquad \hat{X}_{\Sigma,0} = C^0(I, T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}), \qquad \hat{X}_{\Sigma,1} = C^0(I, J_\Sigma X_{\Sigma,0}),$$

where J_Σ is given by (9.2). Then $\hat{X}_\Sigma = \hat{X}_{\Sigma,0} \oplus \hat{X}_{\Sigma,1}$ and we can define the map

$$\begin{aligned} \widehat{F_{\Sigma,(B_0,\Psi_0)}} &: \mathcal{V}_\Sigma \to \hat{X}_{\Sigma,0}\oplus\hat{X}_{\Sigma,1} \\ z = (z_0,z_1) &\mapsto (z_0, z_1 - \widehat{r_\Sigma}\widehat{E^1_{(B_0,\Psi_0)}}(\widehat{P_{(B_0,\Psi_0)}}z_0)), \end{aligned} \tag{10.49}$$

where $\mathcal{V}_\Sigma \subset \hat{X}_\Sigma$ is an open subset containing 0, $E^1_{(B_0,\Psi_0)}$ is defined as in Theorem 4.8, *and* $P_{(B_0,\Psi_0)}$ *is the Poisson operator defined as in Theorem 3.13. For any* $\max(1/2, 2/p) < s' \le s - 1/p$*, we can choose* $\mathcal{V}_\Sigma$ *to contain a* $C^0(I, B^{s',p}(\Sigma))$ *ball, i.e., there exists a* $\delta > 0$*, depending on* $r_\Sigma(B_0, \Psi_0)$*,* s'*, and* p*, such that*

$$\mathcal{V}_\Sigma \supseteq \{z \in \hat{X}_\Sigma : \|z\|_{C^0(I, B^{s',p}(\Sigma))} < \delta\}.$$

Moreover, we can choose δ *such that the following hold:*

(i) We have $\widehat{F_{\Sigma,(B_0,\Psi_0)}}(0) = 0$ *and* $D_0\widehat{F_{\Sigma,(B_0,\Psi_0)}} = \mathrm{id}$*. For* $\mathcal{V}_\Sigma$ *sufficiently small,* $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ *is a local straightening map for* $C^0(I, \mathcal{L}^{s',p})$ *within* $\mathcal{V}_\Sigma$*.*

(ii) The maps $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ *and* $\widehat{F_{\Sigma,(B_0,\Psi_0)}}^{-1}$ *preserve* $B^{s,p}(I \times \Sigma)$ *regularity, i.e. they map* $B^{s,p}(I \times \Sigma)$ *configurations to* $B^{s,p}(I \times \Sigma)$ *configurations. Moreover, the term* $\widehat{r_\Sigma}\widehat{E^1_{(B_0,\Psi_0)}}(\widehat{P_{(B_0,\Psi_0)}})$ *maps* $B^{s,p}(I \times \Sigma)$ *configurations to* $B^{(s,1-1/p-\epsilon),p}(I \times \Sigma)$*, for any* $\epsilon > 0$*.*

(iii) We can choose δ *uniformly for* $r_\Sigma(B_0, \Psi_0)$ *in a sufficiently small* $B^{s',p}(\Sigma)$ *neighborhood of any configuration in* $\mathcal{L}^{s-1/p,p}$*.*

Proof By Lemma 4.14, we know that $F_{\Sigma,(B_0,\Psi_0)}$ is a local straightening map for $\mathcal{L}^{s-1/p,p}$. Based on that lemma, it easily follows that $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ is a local straightening map for $C^0(I, \mathcal{L}^{s-1/p,p})$, thus establishing (i). It is (ii) that mainly needs verification, and this requires some highly nontrivial analysis. We begin by estimating the operators occurring in $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ one by one.

First, we show the boundedness of the slicewise Calderon projection $\widehat{P^+_{(B_0,\Psi_0)}}$ in the $B^{s,p}(I \times \Sigma)$ topology, so that the decomposition $z = (z_0, z_1)$ extends to the $B^{s,p}(I \times \Sigma)$ topology. Here, $P^+_{(B_0,\Psi_0)}$ is the Calderon projection as defined in Theorem 3.13, and it yields a bounded map

$$P^+_{(B_0,\Psi_0)} : \mathcal{T}^{t,p}_\Sigma \to B^{t,p}(T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}), \qquad 0 \le t \le s + 1 - 1/p, \tag{10.50}$$

since $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$. Here, by a slight abuse of notation with regard to Notation 10.2, we define $B^{t,p}(T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p})$ to be the intersection of $T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}$ with $B^{t,p}(\Sigma)$ configurations in case $t \ge s - 1/p$; otherwise, we take the $B^{t,p}(\Sigma)$ closure. The map $P^+_{(B_0,\Psi_0)}$ also extends to a bounded operator on Sobolev spaces

$$P^+_{(B_0,\Psi_0)} : H^{t,p}\mathcal{T}_\Sigma \to H^{t,p}(T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}), \qquad 0 \le t < s + 1 - 1/p, \tag{10.51}$$

where $H^{t,p}(T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p})$ is defined as above, see Remark 4.17. From this, since $P^+_{(B_0,\Psi_0)}$ is time-independent, one can apply Lemma 10.6 to conclude, in particular, that

$$\widehat{P^+_{(B_0,\Psi_0)}} : \mathrm{Maps}^{s,p}(I, \mathcal{T}_\Sigma) \to \mathrm{Maps}^{s,p}(I, T_{r_\Sigma(B_0,\Psi_0)}\mathcal{L}^{s-1/p,p}), \tag{10.52}$$

is bounded. Thus, we have that $\widehat{P^+_{(B_0,\Psi_0)}}$ is a bounded projection from $\hat{X}_\Sigma$ onto $\hat{X}_{\Sigma,0}$ with kernel $\hat{X}_{\Sigma,1}$.

Proceeding as above, we also obtain a bounded slicewise Poisson operator

$$\begin{aligned}\widehat{P_{(B_0,\Psi_0)}} : \mathrm{Maps}^{s,p}(I,\mathcal{T}_\Sigma) &\to \mathrm{Maps}^{(s,1/p),p}(I,T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}) \\ &= \{z \in \mathrm{Maps}^{(s,1/p),p}(I,\mathcal{T}) : z(t) \in T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}, \quad \forall t \in I\}\end{aligned} \tag{10.53}$$

and its range is contained in $C^0(I, L^\infty\mathcal{T})$, due to the embedding

$$\mathrm{Maps}^{(s,1/p),p}(\mathcal{T}) \hookrightarrow C^0(I, B^{s,p}\mathcal{T}) \hookrightarrow C^0(I, L^\infty\mathcal{T}).$$

(Here, when we apply Lemma 10.6 to $\widehat{P_{(B_0,\Psi_0)}}$, it makes no difference that we map configurations on $I \times \Sigma$ to configurations on $I \times Y$, i.e., Lemma 10.6 is unchanged if the domain and range manifolds are different.)

Next, we estimate the mapping properties of $\widehat{E^1_{(B_0,\Psi_0)}}$. Namely, we show that

$$\widehat{E^1_{(B_0,\Psi_0)}} : \mathrm{Maps}^{(s,1/p),p}(I, T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}) \dashrightarrow \mathrm{Maps}^{(s,1),p}(I,\mathcal{T}), \tag{10.54}$$

is bounded, i.e. $\widehat{E^1_{(B_0,\Psi_0)}}$ smooths by $1 - 1/p$ derivatives in the Y directions. Here, we restrict the domain of (10.54) so that it lies inside the set

$$\mathrm{Maps}^{(s,1/p),p}(I,U) = \{z \in \mathrm{Maps}^{(s,1/p),p}(I, T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}) : z(t) \in U\}, \tag{10.55}$$

where $U \subset T_{(B_0,\Psi_0)}\mathcal{M}^{s,p}$, as defined in Theorem 4.8, is a domain on which $E^1_{(B_0,\Psi_0)}$ is defined. This is where our slicewise estimates made in Lemma 10.6 and Corollary 10.7 come into play. By (4.29), we have

$$\widehat{E^1_{(B_0,\Psi_0)}}(z) = -\widehat{Q_{(B_0,\Psi_0)}}\big(\widehat{F_{(B_0,\Psi_0)}}^{-1}(z), \widehat{F_{(B_0,\Psi_0)}}^{-1}(z)\big), \tag{10.56}$$

where $F_{(B_0,\Psi_0)}$ is a local straightening map for $(B_0,\Psi_0) \in \mathcal{M}^{s,p}$, see Lemma 4.6 and Theorem 4.8.

To estimate $\widehat{E^1_{(B_0,\Psi_0)}}$, we first estimate $\widehat{F_{(B_0,\Psi_0)}}^{-1}$. This is the most difficult step of all. We want to use the ideas from Lemma 10.6 to conclude that the smooth time-independent map $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves the $B^{(s,1/p),p}(I \times Y)$ topology. Namely, using the Fubini property, we can write

$$B^{(s,1/p),p}(I \times Y) = L^p(I, B^{s+1/p,p}(Y)) \cap B^{s,p}(I, H^{1/p,p}(Y)). \tag{10.57}$$

Since $(B_0,\Psi_0) \in \mathcal{M}^{s,p}$, the proofs of Lemma 4.7, Theorem 4.8, and Corollary 4.9 show that $F^{-1}_{(B_0,\Psi_0)}$ preserves $B^{t,p}(Y)$ regularity for $1/p \le t \le s+1$, hence for $t = s + 1/p$ in particular. Thus, we have trivially that the time-independent operator $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves $L^p(I, B^{s+1/p,p}(Y))$ regularity. The nontrivial step is to show that $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves

$B^{s,p}(I, H^{1/p,p}(Y))$ regularity. To show this, we want to interpolate between the estimates[8]

$$\widehat{F_{(B_0,\Psi_0)}}^{-1} : H^{k,p}(I, H^{1/p,p}(Y)) \dashrightarrow H^{k,p}(I, H^{1/p,p}(Y)), \qquad k \geq 0, \tag{10.58}$$

where we abuse notation by letting $H^{k,p}(I, H^{1/p,p}(Y))$ denote the closure of $\mathrm{Maps}(I, \mathcal{T})$ in said topology. Here the domain of (10.58) will be specified in a moment.

We want to use interpolation because fractional regularity in time is difficult to grasp; on the other hand, integer Sobolev spaces, such as those appearing in (10.58) are amenable to estimates via the Leibnitz rule. The difficulty of course is that $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ is nonlinear so that Lemma 10.6 does not apply. Additionally, $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ is a highly nonexplicit map, being the slicewise inverse of the operator $F_{(B_0,\Psi_0)}$ and hence defined by the inverse function theorem. In Part I, we were able to estimate $F^{-1}_{(B_0,\Psi_0)}$ (see the proof of Theorem 4.8) because it differed from the identity map by a smoothing operator, but in this case, since we only have smoothing in the space directions, we cannot carry over such a perturbative argument. What saves us in our situation is that there is a theory of interpolation of Lipschitz operators due to Peetre [38]. Moreover, such an interpolation theory extends to operators which are only locally defined (as is the case for $\widehat{F_{(B_0,\Psi_0)}}^{-1}$) under the appropriate conditions. Here, the relevant theorem is Theorem 14.8. We can apply this theorem to $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ by verifying that the following properties hold:

(I) The map $F^{-1}_{(B_0,\Psi_0)}$ is a bounded operator from a subset $\tilde{V}$ of $H^{1/p,p}\mathcal{T}$ into $H^{1/p,p}\mathcal{T}$. (This implies that the map $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ maps $L^\infty(I, \tilde{V})$ into $L^\infty(I, H^{1/p,p}\mathcal{T})$. We then have (10.58) on this domain since $F^{-1}_{(B_0,\Psi_0)}$ is smooth.)

(II) On the same subset $\tilde{V}$ above, the map $F^{-1}_{(B_0,\Psi_0)}$ and all its Fréchet derivatives are Lipschitz. (Hence the corresponding statement is true for $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ on $L^\infty(I, \tilde{V})$ by the Leibnitz rule).

(III) The vector valued Gagliardo-Nirenberg inequality holds: for all integers $0 < m < n$, we have

$$\|\partial_t^m f\|_{L^r(I,\mathcal{X})} \leq \|f\|^{1-\theta}_{L^q(I,\mathcal{X})} \|\partial_t^n f\|^{\theta}_{L^p(I,\mathcal{X})},$$

where $\mathcal{X} = H^{1,p}(Y)$, and r, p, q are any numbers satisfying

$$1 < r, p, q \leq \infty, \quad \frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}, \quad 0 < \theta = m/n < 1.$$

Let us explain why these properties hold. For (I), the statement for $F^{-1}_{(B_0,\Psi_0)}$ holds due to Corollary 4.9. For (II), we use the fact that $F^{-1}_{(B_0,\Psi_0)}$, being the inverse of an analytic (in fact quadratic) map $F_{(B_0,\Psi_0)}$, is itself analytic (i.e. it has a local power series expansion) and hence so is $\widehat{F_{(B_0,\Psi_0)}}^{-1}$. This uses the fact that the inverse function theorem holds in

[8] Here, $H^{k,p}(I, H^{1/p,p}(Y))$ is the same as $W^{k,p}(I, H^{1/p,p}(Y))$ in our present vector-valued function space setting. See Section 13.3.

the analytic category of functions, see e.g. [8]. Consequently, the fact that $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ is analytic means we have estimates for all the derivatives of $F^{-1}_{(B_0,\Psi_0)}$ on some fixed small neighborhood (just as one would have for a holomorphic function of one complex variable, using Cauchy's integral formula or its local power series expansion), which allows us to establish (II). These statements follow from Proposition 21.3.[9] For (III), the case when $\mathcal{X} = \mathbb{R}$ is the classical scalar Gagliardo-Nirenberg inequality. As it turns out, by the work of [43], the general vector valued case holds for *arbitrary* Banach spaces $\mathcal{X}$. Thus (III) holds in particular for $\mathcal{X} = H^{1/p}\mathcal{T}$.

Altogether, from the above, we can apply Theorem 14.8 to our situation, where in that theorem, $\mathcal{Z} = H^{1,p}\mathcal{T}$ and the set U_r is the set $L^\infty(I, \tilde{V})$ for sufficiently small $\tilde{V}$. Here the Lipschitz hypothesis of Theorem 14.8 is satisfied by the same argument as in [38, p. 330], because of (III). Thus, interpolation between the estimates (10.58) shows that

$$\widehat{F_{(B_0,\Psi_0)}}^{-1} : B^{s,p}(I, H^{1/p,p}(Y)) \dashrightarrow B^{s,p}(I, H^{1/p,p}(Y)) \tag{10.59}$$

is bounded, where the domain of (10.59) is the open subset

$$\{z \in \mathrm{Maps}^{(s,1/p),p}(I, \mathcal{T}) : z(t) \in \tilde{V}, \text{ for all } t \in I\} \tag{10.60}$$

of $\mathrm{Maps}^{(s,1/p),p}(I, \mathcal{T})$ with $\tilde{V}$ given by (I) above. Thus, the estimate (10.59) and the trivial estimate that $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves $L^p(I, B^{s+1/p,p}(Y))$ regularity on (10.55) implies that $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves $B^{(s,1/p),p}(I \times Y)$ regularity when restricted to

$$\mathrm{Maps}^{(s,1/p),p}(I, U \cap \tilde{V}). \tag{10.61}$$

Finally, to estimate (10.56), we estimate $\widehat{Q_{(B_0,\Psi_0)}}$ by applying function space multiplication and Corollary 10.7. Since

$$B^{(s,1/p),p}(I \times \Sigma) \hookrightarrow B^{s,p}(I \times \Sigma) \cap L^\infty(I \times \Sigma)$$

by Corollary 13.23, and since the latter space is an algebra by Theorem 13.18, we have the multiplication

$$B^{(s,1/p),p}(I \times \Sigma) \times B^{(s,1/p),p}(I \times \Sigma) \to B^{s,p}(I \times \Sigma) \cap L^\infty(I \times \Sigma).$$

Since

$$\widehat{Q_{(B_0,\Psi_0)}} = \left(\widehat{\mathcal{H}_{(B_0,\Psi_0)}|_{X^{s+1,p}_{(B_0,\Psi_0)}}}\right)^{-1} \widehat{\Pi_{\mathcal{K}^{s,p}_{(B_0,\Psi_0)}}}\,\widehat{\mathfrak{q}} \tag{10.62}$$

[9] In fact, since $\widehat{F_{(B_0,\Psi_0)}}$ is just a quadratic map, one can understand the form of power series expansion of $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ sufficiently well so that one can use avoid the nonlinear interpolation method we have used to establish the boundedness of $\widehat{F_{(B_0,\Psi_0)}}$ on the $B^{(s,1/p),p}(I \times Y)$. Using this method, however, has the unfortunate consequence that the radius of convergence of the power series of $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ (about 0) depends on s and p, and so the set V_Σ of the lemma would depend on s and p (i.e. it may shrink with s and p). This would result in a somewhat awkward proof of Theorem 10.9(iv) and Theorem 12.1 later on. Hence, we adhere to the interpolation method since in some sense it is the "optimal" method, even though for the purposes of Part III, we can get away with other methods that use less machinery.

is quadratic multiplication followed by the operator (10.47), we see that

$$\widehat{Q_{(B_0,\Psi_0)}} : \mathrm{Maps}^{(s,1/p),p}(I \times \mathcal{T}) \times \mathrm{Maps}^{(s,1/p),p}(I \times \mathcal{T}) \to \mathrm{Maps}^{(s,1),p}(I \times \mathcal{T}).$$

Finally, by Theorem 13.22, we have a bounded restriction map

$$\widehat{r_\Sigma} : \mathrm{Maps}^{(s,1),p}(I \times \mathcal{T}) \to \mathrm{Maps}^{(s,1-1/p-\epsilon),p}(I \times \mathcal{T}_\Sigma),$$

where $\epsilon > 0$ is arbitrary.

Altogether, this shows that

$$\widehat{r_\Sigma} \widehat{E^1_{(B_0,\Psi_0)}}(\widehat{P_{(B_0,\Psi_0)}}(z)) \in \mathrm{Maps}^{(s,1-1/p-\epsilon),p}(I \times \mathcal{T}_\Sigma), \qquad z \in \mathcal{V}_\Sigma \tag{10.63}$$

if we choose $\mathcal{V}_\Sigma$ small enough so that $\widehat{P_{(B_0,\Psi_0)}}(\mathcal{V}_\Sigma)$ is contained in the space (10.61). In particular, since $B^{(s,1-1/p-\epsilon),p}(I \times \Sigma) \subset B^{s,p}(I \times \Sigma)$ for ϵ small, altogether, the above estimates and the formula (10.49) show that $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ preserves the $B^{s,p}(I \times \Sigma)$ topology. Since $\widehat{F_{\Sigma,(B_0,\Psi_0)}}^{-1}$ is just given by

$$\widehat{F_{\Sigma,(B_0,\Psi_0)}}^{-1}(z) = (z_0, z_1 - \widehat{r_\Sigma}\widehat{E^1_{(B_0,\Psi_0)}}(\widehat{P_{(B_0,\Psi_0)}} z_0)), \qquad z \in \mathcal{V}_\Sigma,$$

we see that $\widehat{F_{\Sigma,(B_0,\Psi_0)}}^{-1}$ also preserves $B^{s,p}(I \times \Sigma)$ regularity on $\mathcal{V}_\Sigma$. Since the open set U in (10.55) contains an $L^\infty(Y)$ ball by Theorem 4.8, we see that $\mathcal{V}_\Sigma$ can be chosen to contain a $C^0(I, B^{s',p}(\Sigma))$ ball, since

$$\widehat{P_{(B_0,\Psi_0)}} : C^0(I, \mathcal{T}_\Sigma^{s',p}) \to C^0(I, \mathcal{T}^{s'+1/p,p}) \hookrightarrow C^0(I, L^\infty \mathcal{T}) \cap C^0(I, H^{1/p,p}\mathcal{T}).$$

This implies that $\widehat{P_{(B_0,\Psi_0)}}$ maps a small $C^0(I, B^{s',p}(\Sigma))$ ball in $\mathrm{Maps}^{s,p}(I, \mathcal{T}_\Sigma)$ into (10.61).

For (iii), note that the operators

$$\widehat{P_{(B_0,\Psi_0)}} : C^0(I, \mathcal{T}_\Sigma^{s',p}) \to C^0(I, \mathcal{T}^{s'+1/p,p}) \tag{10.64}$$

$$\widehat{F_{(B_0,\Psi_0)}}^{-1} : C^0(I, H^{1/p,p}\mathcal{T}) \dashrightarrow C^0(I, H^{1/p,p}\mathcal{T}) \tag{10.65}$$

vary continuously with $r_\Sigma(B_0, \Psi_0) \in \mathcal{L}^{s-1/p,p}$ in the $B^{s',p}(\Sigma)$ topology, i.e. the above maps vary continuously with $(B_0, \Psi_0) \in \mathcal{M}^{s,p}$ in the $B^{s'+1/p,p}(Y)$ topology. This follows from the work in Part I, which establishes the continuous dependence of all the operators involved in the construction of these operators. From this continuous dependence, we can now deduce that the sets U and $\tilde{V}$ appearing in (10.61) can be constructed locally uniformly in (B_0, Ψ_0). Namely, the set U contains a uniform $L^\infty(Y)$ ball by Theorem 4.8, and set $\tilde{V}$ contains a uniform $H^{1/p,p}(Y)$ ball by Corollary 4.9. Hence $\mathcal{V}_\Sigma$ can be chosen to contain a δ-ball in the $C^0(I, B^{s',p}(\Sigma))$ topology, with δ locally uniform in (B_0, Ψ_0). □

With the above lemmas, we have made most of the important steps in proving our first main theorem in this section. Namely, the above lemma constructs for us chart maps for $\mathrm{Maps}^{s,p}(I, \mathcal{L})$ at constant paths γ_0 via the local straightening map $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$, where γ_0 is

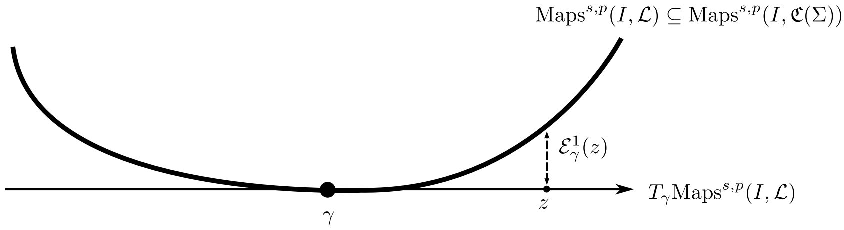

Figure III-1: A local chart for $\mathrm{Maps}^{(s,1/p),p}(I,\mathcal{L})$ at γ. This is the analog of Figure 1 in Part I, for the space of paths through $\mathcal{L}$ in Besov topologies.

identically $r_\Sigma(B_0,\Psi_0)$. Now we want to construct chart maps for $\mathrm{Maps}^{s,p}(I,\mathfrak{C}(\Sigma))$ at an arbitrary path $\gamma \in \mathrm{Maps}^{s,p}(I,\mathcal{L})$. In addition, just as we did for the chart maps at constant paths above, we want to show that the nonlinear portion of the chart map for γ smooths in the Σ directions, and moreover, we want to show the chart map contains "large" domains, i.e., is defined on open neighborhoods of our manifolds with respect to a topology weaker than $B^{s,p}(I\times\Sigma)$. These results will be important for our proofs of Theorems A and B in Section 4. Namely, by having chart maps that smooth, we will be able to gain regularity in the Σ directions for a gauge fixed connection in the proof of Theorem A. By being able to define chart maps on $C^0(I,B^{s',p}(\Sigma))$ neighborhoods of a configuration $\gamma \in \mathrm{Maps}^{s,p}(I,\mathcal{L})$, for some $s'<s$, we will be able to place a sequence of configurations that converge weakly to γ in the $B^{s,p}(I\times\Sigma)$ topology (hence strongly to γ in the $B^{s',p}(I\times\Sigma)$ topology) in the range of a chart map at γ. This will be key when we study sequences of configurations in the proof of compactness in Theorem B. We have the following theorem:

Theorem 10.9 *(Besov Regularity Paths through $\mathcal{L}$) Let I be a bounded interval and let $s>\max(3/p,1/2+1/p)$.*

(i) Then $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ is a closed submanifold of $\mathrm{Maps}^{s,p}(I,\mathfrak{C}(\Sigma))$.

(ii) For any $\gamma\in\mathrm{Maps}^{s,p}(I,\mathcal{L})$, there exists a neighborhood $\mathcal{U}$ of $0\in T_\gamma\mathrm{Maps}^{s,p}(I,\mathcal{L})$ and a smoothing map

$$\mathcal{E}^1_\gamma:\mathcal{U}\to\mathrm{Maps}^{(s,1-1/p-\epsilon),p}(I,\mathcal{T}_\Sigma),$$

where $\epsilon>0$ is arbitrary, such that the map

$$\begin{aligned}\mathcal{E}_\gamma:\mathcal{U}&\to\mathrm{Maps}^{s,p}(I,\mathcal{L})\\ z&\mapsto\gamma+z+\mathcal{E}^1_\gamma(z)\end{aligned}$$

is a diffeomorphism onto a neighborhood of γ in $\mathrm{Maps}^{s,p}(I,\mathcal{L})$.

(iii) For any $\max(2/p,1/2)<s'\le s-1/p$, we can choose both $\mathcal{U}$ and $\mathcal{E}_\gamma(\mathcal{U})$ to contain open $C^0(I,B^{s',p}(\Sigma))$ neighborhoods of $0\in T_\gamma\mathrm{Maps}^{s,p}(I,\mathcal{L})$ and $\gamma\in\mathrm{Maps}^{s,p}(I,\mathcal{L})$ respectively, i.e., there exists a $\delta>0$, depending on γ, s', and p, such that

$$\begin{aligned}\mathcal{U}&\supseteq\{z\in T_\gamma\mathrm{Maps}^{s,p}(I,\mathcal{L}):\|z\|_{C^0(I,B^{s',p}(\Sigma))}<\delta\}\\ \mathcal{E}^1_\gamma(\mathcal{U})&\supseteq\{\gamma'\in\mathrm{Maps}^{s,p}(I,\mathcal{L}):\|\gamma'-\gamma\|_{C^0(I,B^{s',p}(\Sigma))}<\delta\}.\end{aligned}$$

We can choose δ uniformly for all γ in a sufficiently small $C^0(I, B^{s',p}(\Sigma))$ neighborhood of any configuration in $\mathrm{Maps}^{s,p}(I, \mathcal{L})$.

(iv) *Smooth paths are dense in* $\mathrm{Maps}^{s,p}(I, \mathcal{L})$.

Proof The local straightening map $F_{\Sigma}, \widehat{(B_0, \Psi_0)}$ in Lemma 10.8 yields for us induced chart maps in the case when γ is a constant path (see Definition 20.4). From this, to obtain a chart map centered at a general configuration γ, we divide up I as union $I = \cup_{j=0}^n I_j$ of finitely many smaller overlapping subintervals $I_j = [a_j, b_j]$, $a_j < a_{j+1} < b_j$ for $0 \leq j \leq n-1$, and take a constant path γ_j on I_j near $\gamma|_{I_j}$ for which to use as a chart map for $\gamma|_{I_j}$ in $\mathrm{Maps}^{s,p}(I_j, \mathcal{L})$. (We will see why we want overlapping intervals in a bit.) This is possible because the chart maps at constant paths contain $C^0(I, B^{s',p}(\Sigma))$ neighborhoods, and on the small time intervals I_j, an arbitrary path in $\mathrm{Maps}^{s,p}(I_j, \mathcal{L})$ can be approximated in $C^0(I_j, B^{s-1/p,p}(\Sigma)) \hookrightarrow C^0(I_j, B^{s',p}(\Sigma))$ by constant paths. Here, it is important that these neighborhoods depend locally uniformly on the configuration in $C^0(I_j, B^{s',p}(\Sigma))$ by Lemma 10.8, and that we have the embedding $\mathrm{Maps}^{s,p}(I, \mathcal{L}) \to C^0(I, \mathcal{L}^{s-1/p,p})$.

With the above considerations, let γ be a constant smooth path identically equal to $u_0 \in \mathcal{L}$. Then the map $\mathcal{E}_\gamma$ is the chart map associated to $F_{\widehat{\Sigma,(B_0,\Psi_0)}}$ via Lemma 10.8, where $(B_0, \Psi_0) \in \mathcal{M}$ is any configuration satisfying $r_\Sigma(B_0, \Psi_0) = u_0$. Namely, define

$$\mathcal{U} := \mathcal{V}_\Sigma \cap T_\gamma \mathrm{Maps}^{s,p}(I, \mathcal{L}),$$

and then define

$$\begin{aligned} \mathcal{E}_\gamma : \mathcal{U} &\to \mathrm{Maps}^{s,p}(I, \mathcal{L}) \\ z &\mapsto \gamma + F_{\widehat{\Sigma,(B_0,\Psi_0)}}^{-1}(z). \end{aligned} \tag{10.66}$$

The proof of Lemma 10.8 shows that $\mathcal{U}$ contains a $C^0(I, B^{s',p}(\Sigma))$ ball for $\max(2/p, 1/2) < s' \leq s - 1/p$. Furthermore, the map

$$\mathcal{E}^1_\gamma(z) := F_{\widehat{\Sigma,(B_0,\Psi_0)}}^{-1}(z) - z, \tag{10.67}$$

which is therefore the nonlinear part of $F_{\widehat{\Sigma,(B_0,\Psi_0)}}^{-1}(z)$ has the desired mapping properties, since it is simply the map

$$\mathcal{E}^1_\gamma(z) = \widehat{r_\Sigma} \widehat{E^1_{(B_0,\Psi_0)}}(\widehat{P_{(B_0,\Psi_0)}}(\cdot)) : \mathrm{Maps}^{s,p}(I, \mathcal{T}_\Sigma) \dashrightarrow \mathrm{Maps}^{(s,1-1/p-\epsilon),p}(I, \mathcal{T}_\Sigma). \tag{10.68}$$

Altogether, the map (10.66) yields the desired chart map for $\mathrm{Maps}^{s,p}(I, \mathcal{L})$ for the constant path γ, since the map $F_{\widehat{\Sigma,(B_0,\Psi_0)}}$ is a local straightening map for γ

We now consider a general nonconstant, nonsmooth path $\gamma \in \mathrm{Maps}^{s,p}(I, \mathcal{L})$. As we explained, Lemma 10.8 implies that if we choose $a_1 > a_0$ small enough, then on $I_0 = [a_0, a_1]$, the path $\gamma|_{I_0} \in \mathrm{Maps}^{s,p}(I_0, \mathcal{L})$ lies within the range of a chart map for $C^0(I, \mathcal{L}^{s-1/p,p})$ at the constant path identically equal to $\gamma(a_0)$. By the continuous dependence of the size of the chart map in the $C^0(I, B^{s-1/p,p}(\Sigma))$ and since smooth configurations are dense in $\mathcal{L}^{s-1/p,p}$ by Theorem 4.15, we can choose a constant smooth path γ_0 that remains $C^0(I, B^{s-1/p,p}(\Sigma))$ near $\gamma(a_0)$, and such that the chart map $\mathcal{E}_{\gamma_0}$ for γ_0, constructed as above, contains γ in its image. Thus, we have

$$\begin{aligned}\gamma|_{I_0} &= \mathcal{E}_{\gamma_0}(z_0^*)\\ &= \gamma_0 + z_0^* + \mathcal{E}^1_{\gamma_0}(z_0^*), \qquad z_0^* \in T_{\gamma_0}\mathrm{Maps}^{s,p}(I_0,\mathcal{L})\end{aligned}$$

for some z_0^*. Here, to conclude that $z_0^* \in T_{\gamma_0}\mathrm{Maps}^{s,p}(I_0,\mathcal{L})$ has the same regularity as $\gamma|_{I_0}$, we used that γ_0 is smooth so that $\gamma|_{I_0} - \gamma_0 \in \mathrm{Maps}^{s,p}(I_0,\mathcal{T}_\Sigma)$, and then we used that $\widehat{F_{\Sigma,(B_0,\Psi_0)}}^{-1}$ preserves the $B^{s,p}(I\times\Sigma)$ topology by Lemma 10.8.

Of course, we can continue the above process, whereby we have constant smooth paths $\gamma_j \in \mathrm{Maps}(I_j,\mathcal{L})$, and $\gamma|_{I_j}$ is in the image of the chart map $\mathcal{E}_{\gamma_j}$ for γ_j, i.e.,

$$\begin{aligned}\gamma|_{I_j} &= \mathcal{E}_{\gamma_j}(z_j^*)\\ &= \gamma_j + z_j^* + \mathcal{E}^1_{\gamma_j}(z_j^*), \qquad z_j^* \in T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L})\end{aligned}$$

for some z_j^*. In this way, we see that concatenating all the local chart maps $\mathcal{E}_{\gamma_j}$ for the constant smooth paths γ_j, we can define a map

$$\begin{aligned}\mathcal{E}_\gamma : \oplus_{j=0}^n T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L}) &\dashrightarrow \times_{j=0}^n \mathrm{Maps}^{s,p}(I_j,\mathfrak{L})\\ (z_j)_{j=0}^n &\mapsto \Big(t \mapsto \mathcal{E}_{\gamma_j}(z_j^* + z_j)\Big)_{j=0}^n, \quad t\in I_j. \qquad (10.69)\end{aligned}$$

Here, the domain of $\mathcal{E}_\gamma$ is of course restricted to the direct sum of the domains of the individual $\mathcal{E}_{\gamma_j}$.

To get an actual chart map, we must restrict the domain of $\mathcal{E}_\gamma$ above so that its image under $\mathcal{E}_\gamma$ gives an honest path in $\mathrm{Maps}^{s,p}(I,\mathfrak{L})$ when we concatenate all the local paths on the I_j. Thus, define $\mathcal{U}$ by

$$\mathcal{U} \subset \Big\{(z_j) \in \oplus_{j=0}^n T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L}) \ : \ \mathcal{E}_{\gamma_j}(z_j)\big|_{I_j\cap I_{j+1}} = \mathcal{E}_{\gamma_{j+1}}(z_{j+1})\big|_{I_j\cap I_{j+1}}, \quad 0\le j\le n-1\Big\}, \qquad (10.70)$$

where $\mathcal{U}$ is any sufficiently small open subset containing 0 on which $\mathcal{E}_\gamma$ is defined. In this way, we see that the map (10.69) induces a well-defined map

$$\begin{aligned}\mathcal{E}_\gamma : \mathcal{U} &\to \mathrm{Maps}^{s,p}(I_j,\mathcal{L})\\ (z_j)_{j=0}^n &\mapsto \Big(t\mapsto \mathcal{E}_{\gamma_j}(z_j^*+z_j)\Big), \quad t\in I_j. \qquad (10.71)\end{aligned}$$

which maps 0 to γ and the open set $\mathcal{U}$ diffeomorphically onto a neighborhood of $\gamma \in \mathrm{Maps}^{s,p}(I,\mathcal{L})$. Moreover, we can choose $\mathcal{U}$ so that it contains a $C^0(I,B^{s',p}(\Sigma)$ open ball, in which case, $\mathcal{E}_\gamma(\mathcal{U})$ contains a $C^0(I,B^{s',p}(\Sigma))$ neighborhood of γ in $\mathrm{Maps}^{s,p}(I,\mathfrak{L})$.

Note that in defining $\mathcal{E}_\gamma$ as above, while the tangent space $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$ naturally sits inside $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$ as the subspace[10]

$$T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L}) = \{z \in \mathrm{Maps}^{s,p}(I,\mathcal{T}_\Sigma) : z(t) \in T_{\gamma(t)}\mathcal{L}^{s-1/p,p}, \quad \text{for all } t \in I\}, \tag{10.72}$$

we really study this space under the identification

$$T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L}) \cong \Big\{ (z_j)_{j=0}^n \in \Big(\oplus_{j=0}^n T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L}) \Big) : \\ (D_{z_j^*}\mathcal{E}_{\gamma_j})(z_j)\big|_{I_j\cap I_{j+1}} = (D_{z_{j+1}^*}\mathcal{E}_{\gamma_{j+1}})(z_{j+1})\big|_{I_j\cap I_{j+1}}, \quad 0 \le j \le n-1 \Big\}. \tag{10.73}$$

The space (10.72) is difficult to study because it is described as a family of varying subspaces, which makes it hard to understand, for example, the mapping properties of projections onto this space. Indeed, any resulting projection would be time-dependent, but not in a smooth way, since $\gamma \in \mathrm{Maps}^{s,p}(I,\mathcal{L})$ is in general not smooth. Consequently, it would be hard to prove an analog of Lemma 10.6 for such a nonsmooth time-dependent slicewise operator. On the other hand, with (10.73), we understand each factor $T_{\gamma_j}\mathrm{Maps}^{s,p}(I,\mathcal{L})$ from Lemma 10.8. These spaces do have bounded projections onto them, given by a slicewise Calderon projection as in (10.52), which we understand because it is time independent. Thus, in (10.73), we have simplified matters by constructing "local trivializations" of the space $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$. Indeed, (10.73) tells us that on a small time interval I_j, we can identify $T_{\gamma|_{I_j}}\mathrm{Maps}^{s,p}(I_j,\mathcal{L})$ with $T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L})$, and the total space $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$ is obtained by gluing together these local spaces. (This is why we chose the intervals I_j in $I = \cup_{j=0}^n I_j$ to overlap.) When performing estimates for chart maps, it is thus convenient to work with the identification (10.73), whereas when we wish to regard $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$ as a natural subspace of $\mathrm{Maps}^{s,p}(I,\mathfrak{C}(\Sigma))$, then we have the equality (10.72).

Thus, with the identification (10.73), the map $\mathcal{E}^1_\gamma$ is just the concatenation of all the local $\mathcal{E}^1_{\gamma_j}$ and thus has the requisite smoothing properties. One could also compute $\mathcal{E}^1_\gamma$ in the case where we regard $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$ as (10.72), but this will not be necessary. Altogether, we have proven (i) and (ii). Statement (iii) now follows directly from the corresponding property for chart maps at constant paths.

For (iv), the last statement follows from the fact that smooth configurations are dense in $T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L})$. Indeed, since $\gamma_j \in \mathrm{Maps}(I,\mathcal{L})$ is constant and smooth, it has a lift to a constant smooth path in $\mathrm{Maps}(I,\mathcal{M})$, which is identically $(B_j,\Psi_j) \in \mathcal{M}$ for some (B_j,Ψ_j). This follows from Theorem 4.13. Since (B_j,Ψ_j) is smooth, the time-independent slicewise Calderon projection $\widehat{P^+_{(B_j,\Psi_j)}}$ gives a projection of the space of smooth paths $\mathrm{Maps}(I_j,\mathcal{T}_\Sigma)$ onto $T_{\gamma_j}\mathrm{Maps}(I_j,\mathcal{L})$, see Theorem 3.13. If we mollify $z_j^* \in T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L})$ on $I_j \times \Sigma$, we get elements $z_j^\epsilon \in \mathrm{Maps}(I_j,\mathcal{T}_\Sigma)$ such that $z_j^\epsilon \to z_j^*$ in $\mathrm{Maps}^{s,p}(I_j,\mathcal{T}_\Sigma)$ as $\epsilon \to 0$. Hence, we now obtain smooth elements $\widehat{P^+_{(B_0,\Psi_0)}}(z_j^\epsilon)$ belonging to $T_{\gamma_j}\mathrm{Maps}(I_j,\mathcal{L})$ that converge to z_j^* in $T_{\gamma_j}\mathrm{Maps}^{s,p}(I_j,\mathcal{L})$ as $\epsilon \to 0$. We then have that the $\mathcal{E}_{\gamma_j}(z_j^\epsilon)$ are smooth paths which approach $\gamma|_{I_j}$ as $\epsilon \to 0$ (indeed, since γ_j is smooth, one can see from the proof of Lemma 10.8

[10] Note that the tangent space $T_\gamma \mathrm{Maps}^{s,p}(I,\mathcal{L})$ only makes sense since we can prove that $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ is a submanifold of $\mathrm{Maps}^{s,p}(I,\mathfrak{C}(\Sigma))$. Otherwise, (10.72) would just be a formal equality instead of an actual equality.

that $\widehat{F_{\Sigma,\gamma_j(a_j)}}$ preserves C^∞ smoothness, and hence so does $\mathcal{E}_{\gamma_j}$). Gluing together all these paths on the I_j yields a smooth path in $\mathrm{Maps}^{s,p}(I,L)$ approximating γ_j in the $B^{s,p}(I\times\Sigma)$ topology. □

Because of the regularity preservation property in Lemma 10.8(ii) and because we have the mixed regularity estimates in Corollary 10.7(ii), whereby if the base configuration $(B_0,\Psi_0)\in\mathcal{M}^{s_1+s_2,2}$ is more regular we obtain the additional mapping property (10.48), we also get the following theorem for mixed regularity paths through $\mathcal{L}$. The reason we consider these mixed topologies is because they will arise in the bootstrapping procedure occurring in the proof of Theorem A. We only need the below result for small s_2, but we state it for general $s_2\geq 0$.

Theorem 10.10 *(Mixed Regularity Paths through $\mathcal{L}$) Assume the hypotheses of Theorem 10.9. In addition, assume $s_1\geq 3/2$ and $s_2\geq 0$.*

(i) Then $\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$ is a closed submanifold of the space $\mathrm{Maps}^{s,p}(I,\mathfrak{C}(\Sigma))\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathfrak{C}(\Sigma))$.

(ii) For any $\gamma\in\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$, there exists a neighborhood $\mathcal{U}$ of $0\in T_\gamma\Big(\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})\Big)$ and a smoothing map

$$\mathcal{E}^1_\gamma:\mathcal{U}\to\mathrm{Maps}^{(s,1-1/p-\epsilon),p}(I,\mathcal{T}_\Sigma)\cap\mathrm{Maps}^{(s_1,s_2+1-\epsilon'),2}(I,\mathcal{T}_\Sigma),$$

where $\epsilon,\epsilon'>0$ are arbitrary, such that the map

$$\begin{aligned}\mathcal{E}_\gamma:\mathcal{U}&\to\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})\\ z&\mapsto\gamma+z+\mathcal{E}^1_\gamma(z)\end{aligned}$$

is a diffeomorphism onto a neighborhood of γ in $\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$. If $s_1>3/2$ or $s_2>0$, we can take $\epsilon'=0$ above.

(iii) For any $\max(s_2,2/p,1/2)<s'\leq s-1/p$, we can choose both $\mathcal{U}$ and $\mathcal{E}_\gamma(\mathcal{U})$ to contain open $C^0(I,B^{s',p}(\Sigma))$ neighborhoods of $0\in T_\gamma\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$ and $\gamma\in\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$ respectively, i.e., there exists a $\delta>0$, depending on γ, s', s_2, and p such that

$$\begin{aligned}\mathcal{U}&\supseteq\{z\in T_\gamma(\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})):\|z\|_{C^0(I,B^{s',p}(\Sigma))}<\delta\}\\ \mathcal{E}^1_\gamma(\mathcal{U})&\supseteq\{\gamma'\in\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L}):\|\gamma'-\gamma\|_{C^0(I,B^{s',p}(\Sigma))}<\delta\},\end{aligned}$$

We can choose δ uniformly for all γ in a sufficiently small $C^0(I,B^{s',p}(\Sigma))$ neighborhood of any configuration in $\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$.

(iv) Smooth paths are dense in $\mathrm{Maps}^{s,p}(I,\mathcal{L})\cap\mathrm{Maps}^{(s_1,s_2),2}(I,\mathcal{L})$.

Proof The proof is exactly as the same as in Theorem 10.9 and the steps made in Lemma 10.8, only we have to check that the relevant operators have the right mapping properties when we take into account the new topology we have introduced. First, generalizing

Lemma 10.8, we show that given $(B_0,\Psi_0) \in \mathcal{M}^{s,p} \cap \mathcal{M}^{s_1+s_2,2}$, we obtain a local straightening map $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ for $C^0(I,\mathcal{L}^{s-1/p,p})$ that preserves the $B^{s,p}(I\times\Sigma) \cap B^{(s_1,s_2),2}(I\times\Sigma)$ topology. The same proof of Lemma 10.8(ii), redone taking into account the $B^{(s_1,s_2),2}(I\times\Sigma)$ topology, shows that this is indeed the case. Here, the dependence of δ on s_2 reflects the fact that we now have to interpolate the estimate

$$\widehat{F_{(B_0,\Psi_0)}}^{-1} : H^{k,2}(I, H^{s_2+1/2,2}(Y)) \dashrightarrow H^{k,2}(I, H^{s_2+1/2,2}(Y)), \qquad k \geq 0, \tag{10.74}$$

in addition to (10.74), since we want to show that $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves $\mathrm{Maps}^{(s_1,s_2+1/2),2}(I\times Y)$ regularity. (To minimize notation, in the above and in the rest of this proof, we identify configuration spaces with their function space topologies). The analogous set $\tilde{V}$ we obtain in the proof of Lemma 10.8 is thus a subset of $H^{1/p,p}\mathcal{T} \cap H^{s_2+1/2,2}\mathcal{T}$, and hence depends on s_2. Thus, if we define the set

$$\{z \in \mathrm{Maps}^{(s,1/p),p}(I,\mathcal{T}) \cap \mathrm{Maps}^{(s_1,s_2+1/2),2}(I,\mathcal{T}) : z(t) \in \tilde{V}, \text{ for all } t \in I\} \tag{10.75}$$

analogous to (10.60), then on the domain (10.75), the map $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ preserves $\mathrm{Maps}^{(s_1,s_2+1/2),2}(I\times Y)$ regularity. The analogous interpolation argument as before and Corollary 10.7 shows that $\widehat{F_{\Sigma,(B_0,\Psi_0)}}$ and $\widehat{F_{\Sigma,(B_0,\Psi_0)}}^{-1}$ preserve $\mathrm{Maps}^{(s_1,s_2),2}(I\times\Sigma)$ regularity.

Next, we want $s' > s_2$, since this implies $C^0(I,B^{s',p}(\Sigma)) \hookrightarrow C^0(I,H^{s_2,2}(\Sigma))$ and so that

$$\widehat{P_{(B_0,\Psi_0)}} : C^0(I,B^{s',p}(\Sigma)) \to C^0(I,H^{s_2+1/2,2}(Y))$$

is bounded. This implies that $\widehat{P_{(B_0,\Psi_0)}}$ maps a $C^0(I,B^{s',p}(\Sigma))$ small ball into $C^0(I,\tilde{V})$, and hence into a domain on which $\widehat{F_{(B_0,\Psi_0)}}^{-1}$ is well-defined and preserves regularity. Moreover, we also get the requisite local uniformity of δ with respect to γ, by doing the analogous continuous dependence analysis of Lemma 10.8(iii).

To establish the mapping property of $\mathcal{E}^1_\gamma$, it remains to show that

$$\widehat{r_\Sigma}\widehat{E^1_{(B_0,\Psi_0)}}(\widehat{P_{(B_0,\Psi_0)}}(\cdot)) : B^{s,p}(I\times\Sigma) \cap B^{(s_1,s_2),2}(I\times\Sigma) \to B^{(s_1,s_2+1-\epsilon'),2}(I\times\Sigma). \tag{10.76}$$

for (B_0,Ψ_0) smooth. (Here, (B_0,Ψ_0) is a smooth configuration that is nearby γ on a small interval, which we may take to be I, as in the analysis of the previous theorem.) By the exact same argument as in Theorem 10.9, we have bounded maps.

$$\widehat{P_{(B_0,\Psi_0)}} : B^{s,p}(I\times\Sigma) \to B^{(s,1/p),p}(I\times Y),$$
$$\widehat{P_{(B_0,\Psi_0)}} : B^{(s_1,s_2),2}(I\times\Sigma) \to B^{(s_1,s_2+1/2),2}(I\times Y),$$

for $(B_0,\Psi_0) \in \mathcal{M}$. Next, we show that

$$\widehat{Q_{(B_0,\Psi_0)}} : B^{(s,1/p),p}(I\times Y) \cap B^{(s_1,s_2+1/2),2}(I\times Y) \to B^{(s_1,s_2+3/2-\epsilon'),2}(I\times Y). \tag{10.77}$$

Thus, we need to estimate the operators appearing in equation (10.62). First, using the embedding $B^{(s,1/p),p}(I\times Y) \hookrightarrow L^\infty(Y)$ by Corollary 13.23, we have a quadratic multiplication map

$$\widehat{\mathsf{q}} : \left(B^{(s_1,s_2+1/2),2}(I\times Y)\cap L^\infty(Y)\right)^2 \to B^{(s_1,s_2+1/2-\epsilon'),2}(I\times Y)$$

where $\epsilon' > 0$ is arbitrary (we can take $\epsilon' = 0$ if $s_1 > 3/2$ or $s_2 > 0$). This follows from Theorem 13.18. From this, Corollary 10.7(ii) implies (10.77). Finally, we apply Theorem 13.22, which gives us a bounded restriction map

$$\widehat{r_\Sigma} : B^{(s_1,s_2+3/2-\epsilon'),2}(I\times Y) \to B^{(s_1,s_2+1-\epsilon'),2}(I\times\Sigma).$$

Altogether, this completes the proof of (10.76). The proof of the theorem now follows as in Theorem 10.9. □

Remark 10.11

(i) From now on, for any $\gamma \in \mathrm{Maps}^{s,p}(I,\mathcal{L})$, we will not need precisely which model of $T_\gamma\mathrm{Maps}^{s,p}$ we need, i.e., the subspace model (10.72) or the locally "straightened" model (10.73), since both are equivalent. All that matters is that we have chart maps $\mathcal{E}_\gamma$ and $\mathcal{E}^1_\gamma$ as in Theorems 10.9 and 10.10 which obey the the analytic properties stated. This is will be the case for the proofs of Theorems A and B in Section 4.

(ii) Since we have just shown that $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ is indeed a manifold, for $s > \max(3/p, 1/2 + 1/p)$, then the family of spaces $T_{\gamma(t)}\mathcal{L}^{s-1/p,p}$, $t \in I$, does indeed comprise the tangent space $T_\gamma\mathrm{Maps}^{s,p}(I,\mathcal{L})$ via (10.72). By the density of smooth configurations, we have $\mathrm{Maps}^{s,p}(I,\mathcal{L})$ is the $B^{s,p}(I\times\Sigma)$ closure of the space of smooth paths $\mathrm{Maps}(I,\mathcal{L}) = \{z \in C^\infty(I\times\Sigma) : z(t) \in \mathcal{L}\}$ through the Lagrangian. Thus, (10.72) is the same space as

$$B^{s,p}(I\times\Sigma) \text{ closure of } \{z \in \mathrm{Maps}(I,\mathcal{T}_\Sigma) : z(t) \in T_{\gamma(t)}\mathcal{L}, \quad \text{for all } t \in I\}. \qquad (10.78)$$

In general, if we replace the submanifold $\mathcal{L} \subset \mathfrak{C}(\Sigma)$, with another submanifold $\mathfrak{L} \subset \mathfrak{C}(\Sigma)$, which we suppose, like $\mathcal{L}$, is a Fréchet submanifold of the Frechét affine space $\mathfrak{C}(\Sigma)$, then in general, it may not be the case that (10.78) with $\mathcal{L}$ replaced by $\mathfrak{L}$ gives the true tangent space $T_\gamma\mathrm{Maps}^{s,p}(\mathfrak{L})$. Indeed, $\mathrm{Maps}^{s,p}(\mathfrak{L})$ may not even be a manifold. Of course, if $\mathfrak{L}$ satisfies very reasonable properties (i.e. it is defined by local straightening maps obeying the same formal analytic properties as those of $\mathcal{L}$), then $\mathrm{Maps}^{s,p}(\mathfrak{L})$ will be a manifold and $T_\gamma\mathrm{Maps}^{s,p}(I,\mathfrak{L})$ will coincide with the space in 10.78) with $\mathcal{L}$ replaced with $\mathfrak{L}$. In other words, if we define the space (10.78) to be the *formal tangent space* of $\mathrm{Maps}^{s,p}(I,\mathfrak{L})$ at γ, then under reasonable hypotheses on $\mathfrak{L}$, this space will coincide with the honest tangent space $T_\gamma\mathrm{Maps}^{s,p}(I,\mathfrak{L})$, in the appropriate range of s and p.

In the next section, we will be considering abstract Lagrangian submanifolds $\mathfrak{L} \subset \mathfrak{C}(\Sigma)$. All tangent spaces, therefore, will be constructed formally, in the sense above. Of course, when we specialize to $\mathfrak{L} = \mathcal{L}$ a monopole Lagrangian, there is no distinction.

11 Linear Estimates

Based on the results of the previous section, we know that the linearization of (9.7) is well-defined. Indeed, by Theorem 10.9, $\mathrm{Maps}^{1-1/p,p}(\mathbb{R},\mathcal{L})$ is a manifold for $p > 4$. Thus, in this section, as a preliminary step towards proving our main theorems in the next section, we will study the linearized Seiberg-Witten equations, which for simplicity, we consider about a smooth configuration. Since smooth configurations are dense in $\mathrm{Maps}^{1-1/p,p}(\mathbb{R},\mathcal{L})$, we will see that there is no harm in doing so.

In short, our goal is to show that the linearization of (9.7) about a smooth configuration in a suitable gauge makes the problem elliptic. The corresponding elliptic estimates we obtain for the linearized equations will be important in studying the nonlinear equations (9.7) in the next section. For now, we work abstractly and take $\mathfrak{L}$ to be an *arbitrary* Lagrangian submanifold of $\mathfrak{C}(\Sigma)$. Along the way, we will see what kinds of properties such a Lagrangian should possess in order for the linearized equations to be well-behaved, i.e., the associated linearized operator of the equations is Fredholm when acting between suitable function spaces (including anisotropic spaces) and satisfies an elliptic estimate. At the end of this section, we show in Theorem 11.7 that our monopole Lagrangians obey all such properties. This shows that the Seiberg-Witten equations with monopole Lagrangians are well-behaved at the linear level. Nevertheless, by working with abstract Lagrangians, not only do we isolate the essential properties of monopole Lagrangians, but we also leave room for the possibility of generalizing our results to other Lagrangians that obey suitable properties but which are not monopole Lagrangians. (Note that in working with abstract Lagrangians, when we consider their tangent spaces, we do so formally, in the sense of Remark 10.11(ii).)

Altogether, our main results in this section can be roughly described as follows. Here, we replace the time interval $\mathbb{R}$ for our equations with S^1, so that we do not have to worry about issues dealing with asymptotic behavior at infinity. Since our main results, Theorems A and B, are of a local in time nature, there is no harm in working in a compact setting as we will see in their proofs. Moreover, it is clear that all the results of Section 2 carry over verbatim to the periodic setting. Our first main result is Theorem 11.2, which tells us that if we consider a path $\gamma(t)$ through our abstract Lagrangian $\mathfrak{L}$, then if the family of tangent spaces $L(t) = T_{\gamma(t)}\mathfrak{L}$ satisfy the hypotheses of Definition 11.1, then the operator (11.26) induced from the family of spaces $L(t)$ is a Fredholm operator between the appropriate spaces and obeys an elliptic estimate. Indeed, this is the relevant operator to consider, since if $\mathfrak{L}$ is a monopole Lagrangian and $\gamma(t) = r_\Sigma(B(t),\Phi(t))$, the operator considered in Theorem 11.2 is precisely the linearized operator associated to (9.7). (Note however, that in Theorem 11.2, we only linearize about a smooth configuration and we only consider $p = 2$ Besov spaces.) In light of Theorem 11.7, which tells us in particular that monopole Lagrangians satisfy the hypotheses of Theorem 11.2, we have Theorem 11.8, which tells us that the operator associated to linearized Seiberg-Witten equations with monopole Lagrangian boundary conditions is a Fredholm operator.

Our second result concerns the analog of Theorem 11.2 in the anisotropic setting. Whereas Theorem 11.2 is global, in the sense that it holds on all of $S^1 \times Y$, for the anisotropic setting, we work only in a collar neighborhood of the boundary of $S^1 \times Y$, namely $S^1 \times [0,1] \times \Sigma$. This is because the anisotropic spaces we consider will be those that have extra regularity in the Σ directions, and so we must restrict ourselves near the boundary where there is a splting of the underlying space into the Σ directions and the remaining $S^1\times[0,1]$ directions. In fact, when we prove Theorems A and B, we will only need to worry about what happens near the boundary, since the Seiberg-Witten equations are automatically elliptic in the interior modulo gauge. The anisotropic estimates we establish for the linearized Seiberg-Witten equations in the neighborhood $S^1\times[0,1]\times\Sigma$ will allow us to gain regularity for the *nonlinear* Seiberg-Witten equations in the Σ direction. In short, this is because the nonlinear part of the Lagrangian boundary condition smooths in the Σ directions, thanks to Theorems 10.9 and 10.10, and hence the nonlinearity arising from the boundary condition appears only as a lower order term. Using the linear anisotropic estimates of Theorem 11.6 and Corollary 11.9, this extra smoothness in the Σ directions at the boundary allows us to gain regularity in the full neighborhood $S^1\times[0,1]\times\Sigma$ of the boundary (again, only in the Σ directions). This step (which is Step Two in Theorem 12.1) will be key in the next section.

Having described our main results, we give a brief roadmap of this section. In the first part of this section, we describe the appropriate gauge fixing for our linearized equations. We end up with an operator of the form $\frac{d}{dt}+D(t)$, where $D(t)$ is a time-dependent self-adjoint operator. The Fredholm properties of such operators are well-understood, and we want to adapt these known methods to our situation. In the second part, using the same ansatz as before, we then generalize our results to the anisotropic setting. Here, some non-trivial work must be done since the presence of anisotropy is a rather nonstandard situation. In particular, a key result we need to establish is that the resolvent of a certain self-adjoint operator satisfies a decay estimate on anisotropic spaces, see (11.60).

There are two natural choices of gauge for the equations (9.7) that will make them elliptic. Recall that a gauge transformation $g\in\mathrm{Maps}(S^1\times Y,S^1)$ acts on a configuration via

$$g^*(A,\Phi):=(A-g^{-1}dg,g\Phi). \tag{11.1}$$

The first choice of gauge is to find a nearby smooth configuration (A_0,Φ_0) and find a gauge transformation g such that $g^*(A,\Phi)-(A_0,\Phi_0)$ lies in the subspace[11]

$$\mathcal{K}_{(A_0,\Phi_0),n}:=\{(a,\phi)\in T_{(A_0,\Phi_0)}\mathfrak{C}(S^1\times Y):-d^*a+i\mathrm{Re}\,(i\Phi_0,\phi)=0,*a|_{S^1\times\Sigma}=0\} \tag{11.2}$$

orthogonal to the tangent space of the gauge orbit through (A_0,Φ_0). While this is the most geometric choice, it is not the most convenient, since such gauge-fixing can only be done locally, i.e., for (A,Φ) near (A_0,Φ_0). The second choice of gauge fixing is to pick a smooth connection A_0 and place (A,Φ) is the Coulomb-Neumann slice through A_0, i.e., pick a gauge in which (A,Φ) satisfies

$$d^*(A-A_0)=0,\qquad *(A-A_0)|_{S^1\times\Sigma}=0. \tag{11.3}$$

For any $A\in\mathcal{A}(S^1\times Y)$, one can find a unique gauge transformation $g\in\mathcal{G}_{\mathrm{id}}(S^1\times Y)$ up to constants, such that g^*A satisfies (11.3). (Indeed, if we write $g=e^\eta$, with $\eta\in\Omega^0(S^1\times Y;i\mathbb{R})$, this involves solving an inhomogeneous Neumann problem for ξ.)

From now on, we assume our smooth solution (A,Φ) to $SW_4(A,\Phi)=0$ is such that

[11] For more details on gauge fixing, see Section 3 of Part I.

A satisfies (11.3) with respect to some A_0 (to be determined later). Picking any smooth spinor Φ_0 on $S^1 \times Y$, then we have the equation

$$SW_4(A, \Phi) - SW_4(A_0, \Phi_0) = -SW_4(A_0, \Phi_0), \tag{11.4}$$

which, via (9.14), is a semilinear partial differential equation in $(A - A_0, \Phi - \Phi_0)$ with a quadratic linearity.

So write $A_0 = B_0(t) + \alpha_0(t)dt$ as given by (9.12), whereby a connection on $S^1 \times Y$ is expressed as a path of connections on Y plus its temporal component, and define

$$b(t) + \xi(t)dt = A - A_0 \tag{11.5}$$

$$\phi(t) = \Phi(t) - \Phi_0(t). \tag{11.6}$$

Hence, we can express the left-hand-side of (11.4) as a quadratic function of $(b(t), \phi(t), \xi(t)) \in \mathrm{Maps}(S^1, \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \oplus \Omega(Y; i\mathbb{R}))$ depending on the reference configuration (A_0, Φ_0). Then (11.4) becomes

$$\left(\frac{d}{dt} + \mathcal{H}_{(B_0(t),\Phi_0(t))}\right)(b, \phi) + (\rho^{-1}(\phi\phi^*)_0 - d\xi, \rho(b)\phi + \xi\phi + \xi\Phi_0 + \alpha_0\phi) = -SW_4(A_0, \Phi_0). \tag{11.7}$$

Observe that the first term arises from linearizing $\frac{d}{dt}(B, \Phi) - SW_3(B, \Phi)$ and the rest are just remaining terms, which are $-d\xi$ and a quadratic function of (b, ϕ, ξ). This is now a semilinear equation in (b, ϕ, ξ) but it is not elliptic. We now add in the Coulomb-Neumann gauge fixing condition to remedy this. The condition (11.3) becomes

$$\frac{d}{dt}\xi - d^*b = 0, \qquad *b|_{S^1 \times \Sigma} = 0. \tag{11.8}$$

If we add this equation to (11.7), then we obtain the system of equations

$$\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_{(B_0(t),\Phi_0(t))}\right)(b, \phi, \xi) = -\left(\rho^{-1}(\phi\phi^*)_0, \rho(b)\phi + \xi\phi + \xi\Phi_0 + \alpha_0\phi, 0\right) - SW_4(A_0, \Phi_0) \tag{11.9}$$

$$*b|_{S^1 \times \Sigma} = 0, \tag{11.10}$$

where for any configuration $(B_0, \Psi_0) \in \mathfrak{C}(Y)$, the operator $\widetilde{\mathcal{H}}_{(B_0,\Phi_0}$ is the augmented Hessian given by

$$\widetilde{\mathcal{H}}_{(B_0,\Psi_0)} = \begin{pmatrix} \mathcal{H}_{(B_0,\Psi_0)} & -d \\ -d^* & 0 \end{pmatrix} : \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R}) \to \mathcal{T} \oplus \Omega^0(Y; i\mathbb{R}). \tag{11.11}$$

Thus, for every t, the operator $\widetilde{\mathcal{H}}_{(B_0(t),\Phi_0(t))}$ augments the original Hessian $\mathcal{H}_{(B_0(t),\Phi_0(t))}$ by taking into account the Coulomb gauge fixing and the additional $-d\xi$ term that appears in (11.7). To simplify the form of the equations (11.13) even a bit more, we can fix a smooth reference connection $B_{\mathrm{ref}} \in \mathcal{A}(Y)$ and consider the time-independent augmented Hessian

$$\widetilde{\mathcal{H}}_0 := \widetilde{\mathcal{H}}_{(B_{\mathrm{ref}},0)}. \tag{11.12}$$

Then we can write (11.9)-(11.10) as the system

$$\left(\frac{d}{dt}+\widetilde{\mathcal{H}}_0\right)(b,\phi,\xi) = N_{(A_0,\Phi_0)}(b,\phi,\xi) - SW_4(A_0,\Phi_0) \tag{11.13}$$

$$*b|_{S^1\times\Sigma} = 0, \tag{11.14}$$

where $N_{(A_0,\Phi_0)}(b,\phi,\xi)$ is the quadratic multiplication map

$$N_{(A_0,\Phi_0)}(b,\phi,\xi) = -\big(\rho^{-1}(\phi\phi^*)_0, \rho(b)\phi+\xi\phi+\xi\Phi_0+\alpha_0\phi, 0\big) - (B_0(t)-B_{\mathrm{ref}}, \Phi_0(t))\#(b,\phi). \tag{11.15}$$

Here and elsewhere, # denotes a bilinear pointwise multiplication map whose exact form is immaterial.

Thus, the equations (11.13) and (11.14) are altogether the Seiberg-Witten equations in Coulomb-Neumann gauge. Observe that (11.13) is a semilinear elliptic equation. Indeed, the left-hand side is a smooth constant coefficient (chiral) Dirac operator[12] $\frac{d}{dt}+\widetilde{\mathcal{H}}_0$ while the right-hand side is a quadratic nonlinearity.

The boundary condition we impose on our configuration (A,Φ) (aside from the Neumann boundary condition arising from gauge-fixing) is that

$$r_\Sigma(B(t),\Phi(t)) \in \mathfrak{L}, \qquad t\in S^1, \tag{11.16}$$

where $\mathfrak{L}\subset\mathfrak{C}(\Sigma)$ is a Lagrangian submanifold. Recall that the symplectic structure on $\mathfrak{C}(\Sigma)$ is given by the constant symplectic form (3.80) on each tangent spaces to $\mathfrak{C}(\Sigma)$. We will see shortly why the Lagrangian property is important. Altogether then, it is the linearization of the full system (11.13), (11.14), and (11.16), that we want to study.

If we linearize the equations (11.13) at a smooth configuration (A,Φ), then we obtain the linear operator

$$\frac{d}{dt}+\widetilde{\mathcal{H}}_0 - D_{(A,\Phi)}N_{(A_0,\Phi_0)} \tag{11.17}$$

acting on the space

$$\mathrm{Maps}(S^1, \Omega^1(Y;i\mathbb{R})\oplus\Gamma(\mathcal{S})\oplus\Omega^0(Y;i\mathbb{R})) = \mathrm{Maps}(S^1,\widetilde{\mathcal{T}}). \tag{11.18}$$

To linearize the boundary condition[13] (11.16), we introduce the following setup. Consider the full restriction map

$$\begin{aligned} r:\widetilde{\mathcal{T}} &\to \widetilde{\mathcal{T}}_\Sigma = \Omega^1(\Sigma;i\mathbb{R})\oplus\Gamma(\mathcal{S}_\Sigma)\oplus\Omega^0(\Sigma;i\mathbb{R})\oplus\Omega^0(\Sigma;i\mathbb{R}) \\ (b,\phi,\xi) &\mapsto (b|_\Sigma, \phi|_\Sigma, -b(\nu), \xi|_\Sigma), \end{aligned} \tag{11.19}$$

mapping $\widetilde{\mathcal{T}}$ to its boundary data $\widetilde{\mathcal{T}}_\Sigma$ on Σ. Here, ν is the unit outward normal to Σ, and so $b(\nu)$ is the normal component of b at the boundary; the rest of the components of r

[12]Indeed, one can check that $\left(\frac{d}{dt}-\widetilde{\mathcal{H}}_0\right)\left(\frac{d}{dt}+\widetilde{\mathcal{H}}_0\right)$ is a Laplace-type operator.

[13]Unlike Section 2, here we work with smooth configurations, so if $\mathfrak{L}$ is a smooth manifold, then $\mathrm{Maps}(S^1,\mathfrak{L})$ is automatically a smooth manifold and it can be linearized in the expected way. In subsequent steps, we will be taking various closures of the tangent spaces to $\mathrm{Maps}(S^1,\mathfrak{L})$ and in this regard, see Remark 10.11(ii).

represent the tangential components of (b, ϕ, ξ) restricted to the boundary. On the space of paths, the map r induces a slicewise restriction map, which by abuse of notation we again denote by r (instead of $\widehat{r}$):

$$r : \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}) \to \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_\Sigma). \tag{11.20}$$

Thus, specifying boundary conditions on the space $\mathrm{Maps}(S^1, \widetilde{\mathcal{T}})$ for the linearized operator (11.17) is equivalent to specifying a subspace of $\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_\Sigma)$ which determines the admissible boundary values.

The linearization of $\mathrm{Maps}(S^1, \mathfrak{L})$ along a path γ through $\mathfrak{L}$ along with the Neumann boundary condition (11.14) will determine for us a subspace of $\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_\Sigma)$. In order to express this, let $L \subset \mathcal{T}_\Sigma$ be any subspace and define the augmented space

$$\begin{aligned} \widetilde{L} &:= L \oplus 0 \oplus \Omega^0(\Sigma; i\mathbb{R}) \\ &\subseteq \mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) = \widetilde{\mathcal{T}}_\Sigma. \end{aligned} \tag{11.21}$$

This subspace determines the subspace

$$\widetilde{\mathcal{T}}_L = \{(b, \phi, \xi) \in \widetilde{\mathcal{T}} : r(b, \phi, \xi) \in \widetilde{L}\} \tag{11.22}$$

of $\widetilde{\mathcal{T}}$ whose boundary values lie in $\widetilde{L} \subset \widetilde{\mathcal{T}}_\Sigma$. Given a family of subspaces $L(t) \subset \mathcal{T}_\Sigma$, $t \in S^1$, we thus get a corresponding family of spaces $\widetilde{L}(t) \subset \widetilde{\mathcal{T}}_\Sigma$ and $\widetilde{\mathcal{T}}_{L(t)} \subset \widetilde{\mathcal{T}}$. The spaces $\widetilde{\mathcal{T}}_{L(t)}$ can be regarded as a family of domains for the operator $\widetilde{\mathcal{H}}_0$, and thus, we can regard the space

$$\begin{aligned} \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{L(t)}) &:= \{(b(t), \phi(t), \xi(t)) \in \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}) : r(b(t), \phi(t), \xi(t)) \in \widetilde{\mathcal{T}}_{L(t)}, \text{ for all } t \in S^1\} \\ &\subset \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}) \end{aligned} \tag{11.23}$$

as a domain for (11.17).

Altogether then, using this setup, we see that the linearization of the Seiberg-Witten equations with boundary conditions (11.16) at a smooth configuration (A, Φ), where A is in Coulomb-Neumann gauge (11.3), yields the operator

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 - D_{(A,\Phi)} N_{(A_0,\Phi_0)} : \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{L(t)}) \to \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}) \tag{11.24}$$

$$L(t) = T_{r_\Sigma(B(t),\Phi(t))} \mathfrak{L}, \qquad t \in S^1. \tag{11.25}$$

Indeed (11.25) is precisely the linearization of (11.16), and this linearized boundary condition along with the Neumann boundary condition (11.14), is precisely what defines the domain $\widetilde{\mathcal{T}}_{L(t)}$. We want to obtain estimates for the operator (11.24) on the appropriate function space completions. For this, it suffices to consider the constant-coefficient operator

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 : \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{L(t)}) \to \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}), \tag{11.26}$$

since (11.26) differs from (11.24) by a smooth multiplication operator for (A, Φ) and (A_0, Φ_0)

smooth.

$$
\begin{array}{ccccc}
\widetilde{\mathcal{T}}_\Sigma & \xleftarrow{\;r\;} & \widetilde{\mathcal{T}} & & \\
\uparrow & & \uparrow & \searrow^{\widetilde{\mathcal{H}}_0} & \\
 & & & & \widetilde{\mathcal{T}} \\
\cup & & \cup & \nearrow_{\widetilde{\mathcal{H}}_0} & \\
\widetilde{L} & \xleftarrow[\;r\;]{} & \widetilde{\mathcal{T}}_L & &
\end{array}
$$

$$
\begin{array}{ccccc}
\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_\Sigma) & \xleftarrow{\;r\;} & \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}) & & \\
\uparrow & & \uparrow & \searrow^{\frac{d}{dt}+\widetilde{\mathcal{H}}_0} & \\
 & & & & \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}) \\
\cup & & \cup & \nearrow_{\frac{d}{dt}+\widetilde{\mathcal{H}}_0} & \\
\mathrm{Maps}(S^1, \widetilde{L}(t)) & \xleftarrow[\;r\;]{} & \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{L(t)}) & &
\end{array}
\tag{11.27}
$$

Let us now make use the requirement that the manifold $\mathfrak{L}$ is Lagrangian submanifold of $\mathfrak{C}(\Sigma)$. What this implies is that each tangent space $L(t) = T_{\gamma(t)}\mathfrak{L}$ to $\mathfrak{L}$ is a Lagrangian subspace of $\mathcal{T}_\Sigma$. Consequently, each augmented space $\widetilde{L(t)}$ is product Lagrangian in the symplectic space $\widetilde{\mathcal{T}}_\Sigma$, where the symplectic form on $\widetilde{\mathcal{T}}_\Sigma$ is given by the product symplectic form

$$
\begin{aligned}
&\tilde{\omega} : \widetilde{\mathcal{T}}_\Sigma \oplus \widetilde{\mathcal{T}}_\Sigma \to \mathbb{R} \\
&\tilde{\omega}((a, \phi, \alpha_1, \alpha_0), (b, \psi, \beta_1, \beta_0)) = \omega((a, \phi), (b, \psi)) + \int_\Sigma (-\alpha_0\beta_1 + \alpha_1\beta_0).
\end{aligned}
\tag{11.28}
$$

Recall from Part I that the symplectic forms ω and $\tilde{\omega}$ naturally arise from Green's formula for Hessian and augmented Hessian operators $\mathcal{H}_{(B,\Psi)}$ and $\widetilde{\mathcal{H}}_{(B,\Psi)}$, respectively, for any $(B, \Psi) \in \mathfrak{C}(Y)$. In this way, each Lagrangian subspace $L(t) \subset \mathcal{T}_\Sigma$ yields for us a domain $\widetilde{\mathcal{T}}_{L(t)}$ on which $\widetilde{\mathcal{H}}_0$ is symmetric, since for all $x, y \in \widetilde{\mathcal{T}}_{L(t)}$, we have

$$
\tilde{\omega}(r(x), r(y)) = -(\widetilde{\mathcal{H}}_0 x, y)_{L^2(Y)} + (x, \widetilde{\mathcal{H}}_0 y)_{L^2(Y)} = 0,
\tag{11.29}
$$

since $r(x), r(y) \in \widetilde{L(t)}$.

Thus, when we impose Lagrangian boundary conditions for the Seiberg-Witten equations, the associated linear operator (11.26) is of the form $\frac{d}{dt} + D(t)$, where $D(t)$ is a formally self-adjoint operator with time-varying domain. In this abstract situation, we have the following. When the domain of $D(t)$ is constant and furthermore, $D(t)$ is a self-adjoint, Fredholm operator (with respect to the appropriate topologies), then there is a vast literature concerning the corresponding operator $\frac{d}{dt} + D(t)$, since the Fredholm and spectral flow properties of such operators, for example, constitute a rich subject. When the domains of $D(t)$ are varying, the results of the constant domain case can be carried over as long as the domains of $D(t)$ satisfy appropriate "trivialization" conditions (e.g., see [42, Appendix A]).

We will study (11.26) from this point of view. There are thus two things we wish to impose on our Lagrangian $\mathfrak{L}$ so that its tangent spaces $L(t) = T_{\gamma(t)}\mathfrak{L}$ all obey the following loosely formulated conditions:

(I) for each domain $\widetilde{\mathcal{T}}_{L(t)}$, the operator $\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}_{L(t)} \to \widetilde{\mathcal{T}}$ is self-adjoint (as opposed to formally self-adjoint) and Fredholm with respect to the appropriate topologies;

(II) the time-varying domains $\widetilde{\mathcal{T}}_{L(t)}$ satisfy the appropriate trivialization conditions.

We now introduce the function spaces we will be considering. We will be working with L^2 spaces, namely the Besov spaces $B^{s,2}$ with exponent $p = 2$. Recall these spaces are also denoted by H^s, though since we have been working primarily with Besov spaces in Part III, we will stick to the notation $B^{s,2}$ to be consistent. We want to work with $p = 2$, because on L^2 spaces, one can employ Hilbert space methods, in particular, one has the spectral theorem and unitarity of the Fourier transform. The $p \neq 2$ analysis developed in the previous section will come into play for the nonlinear analysis of Seiberg-Witten equations, which we take up in the next section. We may thus consider the operator

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 : \mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(t)}) \to \mathrm{Maps}^{k-1,2}(S^1, \widetilde{\mathcal{T}}) \tag{11.30}$$

for all integers $k \geq 1$. The spaces $\mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}})$ is defined as in Definition 10.3. To define the space $\mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(t)})$ with varying domain $\widetilde{\mathcal{T}}_{L(t)}$, one proceed in a similar way for $k \geq 2$. In this case, one can take a trace twice for functions in $B^{k,2}(S^1 \times Y)$, and so $\mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(t)})$ is the subspace of $\mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}})$ whose paths $(b(t), \phi(t), \xi(t))$ satisfy

$$r(b(t), \phi(t), \xi(t)) \in B^{k-1,2}\widetilde{L(t)} \subset \widetilde{\mathcal{T}}_{\Sigma}^{k-1,2}, \qquad t \in S^1.$$

Unfortunately, this definition does not work for $k = 1$. Thus, we have the following definition which works for all $k \geq 1$ and which coincides with the above definition. Namely, we define

$$\mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(t)}) := B^{k,2}(S^1, \widetilde{\mathcal{T}}^{0,2}) \cap L^2(S^1, \widetilde{\mathcal{T}}^{k}_{L(t)}). \tag{11.31}$$

In other words, $\mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(t)})$ consists of those paths that have k time derivatives in the space $L^2(S^1, \widetilde{\mathcal{T}}^{0,2})$ and which belong to $\widetilde{\mathcal{T}}^{k}_{L(t)}$ in the $L^2(S^1)$ sense, where

$$\widetilde{\mathcal{T}}^{k,2}_{L(t)} = \{(b, \psi, \xi) \in \widetilde{\mathcal{T}}^{k,2} : r(b, \psi, \xi) \in B^{k-1/2,2}\widetilde{L(t)} \subset \widetilde{\mathcal{T}}_{\Sigma}^{k-1/2,2}\}$$

makes sense for $k \geq 1$. Thus, we have merely separated variables in (11.31) and ask that all derivatives of order k exist in L^2 and that the appropriate boundary conditions hold.

With the above definitions, we can thus consider the operator (4.15) and the family of operators

$$\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{k+1,2}_{L(t)} \to \widetilde{\mathcal{T}}^{k,2}, \qquad t \in S^1. \tag{11.32}$$

It is these operators and their domains that have to satisfy the appropriate assumptions, by the above discussion, in order for us to obtain suitable estimates for (4.15). According to the first condition (I) above, we want that the family of operators (11.32) to be Fredholm for all $k \geq 0$, and furthermore, for $k = 0$, that they are all self-adjoint. To obtain (II), we make the following definition:

Definition 11.1 Let $L(t)$ be a smoothly varying family of subspaces[14] of $\mathcal{T}_\Sigma$, $t \in S^1$ or $\mathbb{R}$. We say that the $L(t)$ are *regular* if for every $t_0 \in \mathbb{R}$, there exists an open interval $I \ni t_0$ such that for all $t \in I$, there exist isomorphisms $S(t) : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$, satisfying the following properties for all nonnegative integers k:

(i) The map $S(t)$ extends to an isomorphism $S(t) : \widetilde{\mathcal{T}}^{k,2} \to \widetilde{\mathcal{T}}^{k,2}$.

(ii) The map $S(t)$ straightens the family $L(t)$ in the sense that $S(t) : \widetilde{\mathcal{T}}^{k,2}_{L(t_0)} \to \widetilde{\mathcal{T}}^{k,2}_{L(t)}$ is an isomorphism ($k \geq 1$).

(iii) The commutator $[D, S(t)]$, where $D : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ is any first order differential operator, is an operator bounded on $\widetilde{\mathcal{T}}^{k,2}$.

The reason for this definition is that then, on the interval I in the above, where say $t_0 = 0$, the conjugate operator

$$S(t)^{-1}\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_0\right)S(t) : \mathrm{Maps}^{k,2}(I, \widetilde{\mathcal{T}}_{L(0)}) \to \mathrm{Maps}^{k-1,2}(I, \widetilde{\mathcal{T}}) \qquad (11.33)$$

has constant domain, for all $k \geq 1$. Conditions (i) and (ii) ensure that (11.33) is well-defined. Condition (iii) ensures that the conjugate operator (11.33) gives us a lower order perturbation of the original operator, since

$$S(t)^{-1}\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_0\right)S(t) = \frac{d}{dt} + \widetilde{\mathcal{H}}_0 + \left(S(t)^{-1}\frac{d}{dt}S(t) + S^{-1}(t)[\widetilde{\mathcal{H}}_0, S(t)]\right). \qquad (11.34)$$

Thus, (iii) and the fact that $S(t)$ depends smoothly on t implies that the right-most term of the above is a bounded operator. One can now understand the time-varying domain case in terms of the constant domain case via this conjugation (see Theorem 11.2). Thus, the map $S(t)$ trivializes, or "straightens", the family of subspaces $\widetilde{\mathcal{T}}_{L(t)}$, and it is the existence of such an $S(t)$ that expresses precisely what we mean by condition (II) above.

We will show later that when $L(t)$ are the subspaces arising from linearizing a monopole Lagrangian $\mathfrak{L}$ along a smooth path, then the operators (11.32) are Fredholm for all k, self-adjoint for $k = 0$, and the $L(t)$ are regular (see Theorem 11.7). This uses the fundamental analysis concerning monopole Lagrangians in Part I. Assuming these properties hold for some general Lagrangian submanifold $\mathfrak{L} \subset \mathfrak{C}(\Sigma)$, we have our first result concerning the linearization of the Seiberg-Witten equations with (general) Lagrangian boundary conditions:

Theorem 11.2 *(i) Suppose the operators (11.32) are Fredholm for all $k \geq 0$, and that furthermore, they are self-adjoint for $k = 0$. Also, suppose the family of spaces $L(t) = T_{\gamma(t)}\mathfrak{L}$, $t \in S^1$, are regular. Then*

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 : \mathrm{Maps}^{k+1,2}(S^1, \widetilde{\mathcal{T}}_{L(t)}) \to \mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}) \qquad (11.35)$$

[14] See Definition 18.9.

is Fredholm for every $k \geq 0$, and we have the elliptic estimate

$$\|(b,\phi,\xi)\|^2_{B^{k+1,2}(S^1\times Y)} \leq C\left(\left\|\left(\frac{d}{dt}+\widetilde{\mathcal{H}}_0\right)(b,\phi,\xi)\right\|^2_{B^{k,2}(S^1\times Y)} + \|(b,\phi,\xi)\|^2_{B^{k,2}(S^1\times Y)}\right). \tag{11.36}$$

(ii) *(Elliptic regularity) If $(b,\phi,\xi) \in \mathrm{Maps}^{1,2}(S^1,\widetilde{\mathcal{T}}_{L(t)})$ satisfies $(\frac{d}{dt}+\widetilde{\mathcal{H}}_0)(b,\phi,\xi) \in \mathrm{Maps}^{k,2}(S^1,\widetilde{\mathcal{T}})$, then $(b,\phi,\xi) \in \mathrm{Maps}^{k+1,2}(S^1,\widetilde{\mathcal{T}}_{L(t)})$ and it satisfies the elliptic estimate (11.36).*

Proof (i) First, let $k=0$ and suppose $L(t)\equiv L(0)$ is independent of t. Let us make the abbreviations $x=(b,\phi,\xi)$ and $\partial_t = \frac{d}{dt}$. There are two methods to obtain (11.36). The first proceeds as follows. We prove the identity

$$\|(\partial_t+\widetilde{\mathcal{H}}_0)x\|^2_{L^2(S^1\times Y)} = \|\partial_t x\|^2_{L^2(S^1\times Y)} + \|\widetilde{\mathcal{H}}_0 x\|^2_{L^2(S^1\times Y)}. \tag{11.37}$$

via an integration by parts. Here, we can write the cross term of (11.37) as

$$(\partial_t x,\widetilde{\mathcal{H}}_0 x)_{L^2(Y)} + (\widetilde{\mathcal{H}}_0 x,\partial_t x)_{L^2(Y)} = \int_{S^1}\frac{d}{dt}(x,\widetilde{\mathcal{H}}_0 x)_{L^2(Y)}dt, \tag{11.38}$$

because of the self-adjointness of

$$\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{1,2}_{L(t)} \to \widetilde{\mathcal{T}}^{0,2}. \tag{11.39}$$

and the time-independence of $L(t)$. The term (11.38) vanishes since we are integrating an exact form over S^1. Next, since (11.39) is Fredholm by hypothesis, we also have

$$\|x(t)\|^2_{B^{1,2}(Y)} \leq C(\|\widetilde{\mathcal{H}}_0 x(t)\|^2_{L^2(Y)} + \|x(t)\|^2_{L^2(Y)}). \tag{11.40}$$

for every $t\in S^1$. Integrating this estimate over S^1, using this in (11.37), and using the fact that $\|x\|_{B^{1,2}(S^1\times Y)}$ is equivalent to $\|\partial_t x\|_{L^2(S^1\times Y)} + \|x\|_{L^2(S^1,B^{1,2}(Y))}$, we have the elliptic estimate

$$\|x\|^2_{B^{1,2}(S^1\times Y)} \leq C(\|(\partial_t+\widetilde{\mathcal{H}}_0)x\|^2_{L^2(S^1\times Y)} + \|x\|^2_{L^2(S^1\times Y)}). \tag{11.41}$$

This shows that the map (11.35) has closed range and finite dimensional kernel. To show that the cokernel is finite dimensional, we use the fact that we have the following weak regularity estimate (for time-varying domains):

$$y\in \mathrm{Maps}^{0,2}(S^1,\widetilde{\mathcal{T}}) \text{ and } \left(\left(\partial_t+\widetilde{\mathcal{H}}_0\right)x,y\right)=0 \text{ for all } x\in\mathrm{Maps}(S^1,\widetilde{\mathcal{T}}_{L(t)}) \Rightarrow y\in \mathrm{Maps}^{1,2}(S^1,\widetilde{\mathcal{T}}_{L(t)}). \tag{11.42}$$

This is proven in [42, Appendix A]). In light of this, an integration by parts shows that the cokernel of (11.35) is is finite dimensional for $k=0$ (in fact, any k), since the adjoint operator $-\partial_t+\widetilde{\mathcal{H}}_0$ obeys the same estimate (11.41).

There is a second approach to proving (11.40) which generalizes to a more general setting that we will need later. Together with weak regularity (11.42), this proves the Fredholm property of (11.35) for $k=0$. The method we use is to apply the Fourier transform (in $t\in S^1=[0,2\pi]/\sim$) to the time-independent operator $\partial_t+\widetilde{\mathcal{H}}_0$, which means we analyze the operator $i\tau + \widetilde{\mathcal{H}}_0$, for $\tau \in \mathbb{Z}$, in Fourier space.

Without loss of generality, we can suppose the Fredholm operator (11.39) is an invertible operator, which we can always do by perturbing $\widetilde{\mathcal{H}}_0$ by a bounded operator. Indeed, the operator $\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{1,2}_{L(0)} \to \widetilde{\mathcal{T}}^{0,2}$, being a self-adjoint Fredholm operator, has discrete spectrum, and so we can perturb $\widetilde{\mathcal{H}}_0$ by some multiple of the identity to achieve invertibility. By self-adjointness, $i\tau + \widetilde{\mathcal{H}}_0$ is invertible for all $\tau \in \mathbb{R}$. Thus, if we have $\left(\partial_t + \widetilde{\mathcal{H}}_0\right) x = y$ and we want to solve for x, we just solve for the Fourier modes. In other words, we have

$$\hat{x}(\tau) = (i\tau + \widetilde{\mathcal{H}}_0)^{-1}\hat{y}(\tau), \qquad \tau \in \mathbb{Z},$$

where, if $z \in \mathrm{Maps}^{0,2}(S^1, \widetilde{\mathcal{T}})$, we have

$$\hat{z}(\tau) = \int_0^{2\pi} e^{-i\tau t} z(t)dt \in \widetilde{\mathcal{T}}^{0,2}.$$

Thus, by Plancherel's theorem,

$$\begin{aligned} \|\partial_t x\|^2_{L^2(S^1\times Y)} &= \frac{1}{2\pi}\sum_{\tau\in\mathbb{Z}} \|i\tau\hat{x}(\tau)\|^2_{L^2(Y)} \\ &= \frac{1}{2\pi}\sum_{\tau\in\mathbb{Z}} \left\| \left(\frac{i\tau}{i\tau + \widetilde{\mathcal{H}}_0} \right) \hat{y}(\tau) \right\|^2_{L^2(Y)}. \end{aligned} \tag{11.43}$$

From the spectral theorem, we have

$$\|(i\tau - \widetilde{\mathcal{H}}_0)^{-1}\|_{Op(L^2(Y))} = O(\tau^{-1}). \tag{11.44}$$

From (11.43), this implies

$$\|\partial_t x\|_{L^2(S^1\times Y)} \le C\|y\|_{L^2(S^1\times Y)} = C\|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{L^2(S^1\times Y)}, \tag{11.45}$$

which implies

$$\begin{aligned} \|x\|_{B^{1,2}(S^1\times Y)} &\le C(\|\partial_t x\|_{L^2(S^1\times Y)} + \|\widetilde{\mathcal{H}}_0 x\|_{L^2(S^1\times Y)} + \|x\|_{L^2(S^1\times Y)}) \\ &\le C(2\|\partial_t x\|_{L^2(S^1\times Y)} + \|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{L^2(S^1\times Y)} + \|x\|_{L^2(S^1\times Y)}) \\ &\le C'(\|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{L^2(S^1\times Y)} + \|x\|_{L^2(S^1\times Y)}). \end{aligned} \tag{11.46}$$

This establishes the desired elliptic estimate. Thus, this establishes the Fredholm property of (11.35) by previous remarks for $k = 0$.

It remains to consider the general case where $L(t)$ is time-dependent and $k \ge 0$. To control the varying domains, we employ the straightening maps $S(t)$ associated to $L(t)$, as given by Definition 11.1. We reduce to the constant domain case via conjugation by $S(t)$ as in (11.33). For notational convenience, let us suppose the straightening maps can be made periodic, i.e., for all $t \in S^1$, so that we can replace I with S^1 in (11.33). (In general, we have to apply a partition of unity in time and apply local straightening maps on local intervals. We then apply the analysis that is to follow on small time intervals and then sum up the estimates.) By Definition (11.1)(iii), the terms $S(t)\frac{d}{dt}S(t)^{-1}$ and $[\widetilde{\mathcal{H}}(t), S(t)]S(t)^{-1}$ in

(11.33) are bounded on $\mathrm{Maps}^{1,2}(S^1, \widetilde{\mathcal{T}}_{L(0)})$. Thus, we also have the elliptic estimate (11.41) for the conjugate operator (11.33) and hence the original operator on varying domains by isomorphism property, property (ii) of Definition 11.1. This establishes (11.35) for $k = 0$ and for varying domains.

For $k \geq 1$, we proceed inductively in k. Suppose we have established the result for all nonnegative integers up to some $k \geq 0$ and we want to prove it for $k+1$. As in the previous part, we can assume we are in the time-independent case since the time-dependent case reduces to this case via conjugation by $S(t)$. Moreover, by the weak regularity result (11.42), we only need to establish the elliptic estimate (11.36).

So suppose $x \in \mathrm{Maps}^{k+1,2}(S^1, \widetilde{\mathcal{T}}_{L(0)})$. Then $\partial_t x \in \mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(0)})$. By the inductive hypothesis, we have the estimate (11.36) for $\partial_t x$. Now, $\|x\|_{B^{k+1,2}(S^1\times Y)}$ is equivalent to $\|\partial_t x\|_{B^{k,2}(S^1\times Y)} + \|x\|_{L^2(S^1,B^{k+1}(Y))}$. The inductive hypothesis gives us control of the first of these two terms from (11.36); it remains to control the second term $\|x\|_{L^2(S^1,B^{k+1,2}(Y))}$ in order to prove (11.36) for $k+1$. Since $\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{k+1,2}_{L(0)} \to \widetilde{\mathcal{T}}^{k,2}$ is Fredholm, then

$$\|x\|_{L^2(S^1,B^{k+1,2}(Y))} \leq C(\|\widetilde{\mathcal{H}}_0 x\|_{L^2(S^1,B^{k,2}(Y)} + \|x\|_{L^2(S^1\times Y)}). \tag{11.47}$$

Altogether, simple rearrangement yields

$$\begin{aligned}
\|x\|_{B^{k+1,2}(S^1\times Y)} &\sim (\|\partial_t x\|_{B^{k,2}(S^1\times Y)} + \|x\|_{L^2(S^1,B^{k+1}(Y))}) \\
&\leq C(\|\partial_t x\|_{B^{k,2}(S^1\times Y)} + \|\widetilde{\mathcal{H}}_0 x\|_{L^2(S^1,B^{k,2}(Y))} + \|x\|_{L^2(S^1\times Y)}) \\
&\leq C(2\|\partial_t x\|_{B^{k,2}(S^1\times Y)} + \|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{L^2(S^1,B^{k,2}(Y))} + \|x\|_{B^{k,2}(S^1\times Y)}) \\
&\leq 2C(\|(\partial_t + \widetilde{\mathcal{H}}_0)\partial_t x\|_{B^{k-1,2}(S^1\times Y)} + \|\partial_t x\|_{B^{k-1,2}(S^1\times Y)} \\
&\qquad + \|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{L^2(S^1,B^{k,2}(Y))} + \|x\|_{B^{k,2}(S^1\times Y)}) \\
&\leq 2C(\|\partial_t(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{B^{k-1,2}(S^1\times Y)} + \|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{L^2(S^1,B^{k,2}(Y))} \\
&\qquad + \|x\|_{B^{k,2}(S^1\times Y)}) \\
&\sim \left\|(\partial_t + \widetilde{\mathcal{H}}_0)x\right\|_{B^{k,2}(S^1\times Y)} + \|x\|_{B^{k,2}(S^1\times Y)}.
\end{aligned}$$

In the fourth line, we applied the inductive hypothesis to $\partial_t x$. The above computation completes the induction and establishes the elliptic estimate (11.36) for all k.

For (ii), we note that this easily follows from (i), since in proving (i), we implicitly constructed a (left) parametrix for $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ by the above steps. Thus, the a priori elliptic estimate in (i) implies the elliptic regularity statement (ii). □

Remark 11.3 As is usual for elliptic equations, an a priori elliptic estimate implies elliptic regularity as in the above, since there is always a smoothing parametrix implicit in problem. Henceforth, we will always prove a priori elliptic estimates and then state the corresponding elliptic regularity result without additional proof.

11.1 Anisotropic Estimates

While Theorem 11.2 tells us that the linearized operator associated to the Seiberg-Witten equations with suitable Lagrangian boundary conditions is Fredholm, as mentioned in the introduction to this section, we will also need an anisotropic analog. Indeed, we did a great deal of analysis in Section 10 on anisotropic Besov spaces and we will need to generalize the above theorem to such spaces. The reason for this is that in our proof of Theorem A in the next section, we will be bootstrapping the regularity of a configuration in the Σ directions in a neighborhood of the boundary of $\mathbb{R} \times Y$. This boostrapping requires that we gain regularity in certain individual directions from the linearized Seiberg-Witten equations, which means that we want the operator (11.35) to be Fredholm on anisotropic spaces and to have a corresponding elliptic estimate (11.36) on anisotropic spaces.[15]

In detail, on Y we now work in a collar neighborhood $[0,1] \times \Sigma$ of the boundary, and consequently, on $S^1 \times Y$, we work in the collar neighborhood $S^1 \times [0,1] \times \Sigma$. In either case, we let $v \in [0,1]$ denote the inward normal coordinate. We have the corresponding restricted configuration space

$$\widetilde{\mathcal{T}}_{[0,1]\times\Sigma} := \widetilde{\mathcal{T}}|_{[0,1]\times\Sigma} = \Omega^1([0,1] \times \Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}|_{[0,1]\times\Sigma}) \oplus \Omega^0([0,1] \times \Sigma; i\mathbb{R}). \tag{11.48}$$

The restriction map r induces two separate restriction maps

$$r_j : \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}_{\Sigma_j} \tag{11.49}$$

corresponding to the two boundary components

$$\Sigma_j := \{j\} \times \Sigma, j = 0, 1$$

of $[0,1] \times \Sigma$. In this case, we write

$$r = (r_0, r_1) : \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}_{\Sigma_1} \tag{11.50}$$

for the total restriction map. If just write Σ, we will always mean Σ_0.

The space $[0,1] \times \Sigma$ is a product manifold and so we can define anisotropic Besov spaces on it as in Definition 13.21. We have the space $B^{(s_1,s_2),2}([0,1] \times \Sigma)$, the space of functions whose derivatives up to order s_2 in the Σ directions belong to $B^{s_1,2}([0,1] \times \Sigma)$. We define the spaces

$$\widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma} = B^{(k,s),2}\widetilde{\mathcal{T}}_{[0,1]\times\Sigma}$$

of configurations in the $B^{(k,s),2}([0,1] \times \Sigma)$ topology, where $k \geq 0$ is a nonnegative integer and $s \geq 0$. By the anisotropic trace theorem, Theorem 13.22, the restriction maps extend to bounded operators

$$r_j : \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}^{k-1/2+s,2}_{\Sigma_j}, \qquad k \geq 1, \quad j = 0, 1. \tag{11.51}$$

In order to get Fredholm operators mapping between the $\widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma}$ spaces, we need to impose boundary conditions as before, only now we have to impose them on the two boundary

[15] The analysis developed in this section is completely absent in the ASD case, as seen in [54]. Indeed, in the ASD case, there are no spinors, and the structure of the ASD equations alone allows one to easily gain Σ regularity in a collar neighborhood of the boundary of $\mathbb{R} \times Y$ without even using the Lagrangian boundary condition. As a consequence, the hard work we do in Section 10 and in this section, which is to gain Σ regularity near the boundary, is completely unnecessary and absent in the ASD case.

data spaces $\widetilde{\mathcal{T}}^{k-1/2+s,2}_{\Sigma_j}$. For $\widetilde{\mathcal{T}}^{k-1/2+s,2}_{\Sigma_0}$ we impose the same type of boundary condition as before, namely by specifying a subspace L_0 of $\mathcal{T}_{\Sigma_0}$, considering the augmented space $\widetilde{L_0} \subseteq \widetilde{\mathcal{T}}_{\Sigma_0}$, and then taking the $B^{k-1/2+s,2}(\Sigma)$ completion. On $\widetilde{\mathcal{T}}^{k-1/2+s,2}_{\Sigma_1}$ we also choose a subspace as a boundary condition, and for this, we choose any suitable subspace

$$L_1 \subseteq \widetilde{\mathcal{T}}_{\Sigma_1}$$

as follows.

Define the restricted configuration space

$$\widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma,L_0,L_1} = \{x \in \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma} : \; r_0(x) \in B^{k-1/2+s,2}\widetilde{L_0}, \quad r_1(x) = B^{k-1/2+s,2}L_1\},$$

of configurations in $\widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma}$ whose boundary data on $\widetilde{\mathcal{T}}^{k-1/2+s,2}_{\Sigma_0}$ and $\widetilde{\mathcal{T}}^{k-1/2+s}_{\Sigma_1}$ lie in $\widetilde{L_0}$ and L_1, respectively. Ultimately, we want the operator

$$\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{(k+1,s),2}_{[0,1]\times\Sigma,L_0,L_1} \to \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma} \tag{11.52}$$

to be Fredholm, for $k \geq 0$ and $s \geq 0$. The determining of which subspaces L_0 and L_1 determine a Fredholm operator for $\widetilde{\mathcal{H}}_0$ falls within the study of elliptic boundary value problems which we describe in Part IV. From Theorem 15.23, we have the following result: If the subspace on the boundary which determines the boundary condition for $\widetilde{\mathcal{H}}_0$, in this case the subspace

$$B^{k+1/2+s,2}(\widetilde{L_0} \oplus L_1) \subseteq \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1}, \tag{11.53}$$

is such that it is Fredholm with $r(\ker \widetilde{\mathcal{H}}_0)$ (see Definition 18.4), then the associated operator (11.52) is Fredholm. Indeed, when $\widetilde{L_0} \oplus L_1$ is the range of a pseudodifferential operator, this follows from the standard theory of pseudodifferential elliptic boundary conditions. For us however, the space $\widetilde{L_0}$, being a tangent space to $\mathfrak{L}$, a monopole Lagrangian, is only "nearly" pseudodifferential (see Theorem 3.13(ii)), and so we use the more general framework of Theorem 15.23.

In fact, we can say more. Recall from Part I that $r(\ker \widetilde{\mathcal{H}}_0) = r(\ker \widetilde{\mathcal{H}}_{(B_{\mathrm{ref}},0)})$ is given by the range of a zeroth order pseudodifferential operator, the Calderon projection

$$\widetilde{P}_0^+ := \widetilde{P}^+_{(B_{\mathrm{ref}},0)} : \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1} \circlearrowleft . \tag{11.54}$$

Because the symbol of $\widetilde{P}_0^+$ is determined locally by the symbol of $\widetilde{\mathcal{H}}_0$, we see that on Σ_0, the principal symbol of $\widetilde{P}_0^+$ coincides with that of $\Pi^+ : \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0} \to \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0}$, the positive spectral projection of the tangential boundary operator (see Definition 3.6) associated to $\widetilde{\mathcal{H}}_0$ on Σ_0. Identifying $\widetilde{\mathcal{T}}_{\Sigma_0}$ with $\widetilde{\mathcal{T}}_{\Sigma_1}$, since the restriction map $r_1 : \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}_{\Sigma_1}$ is defined as in (11.19), with $-\nu$ now the *outward* normal to Σ_1 (as opposed to $-\nu$ being the inward normal at Σ_0), the principal symbol of $\widetilde{P}_0^+$ on Σ_1 coincides with $\Pi^- : \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1} \to \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1}$, the negative spectral projection of the tangential boundary operator. Indeed, choosing the opposite choice of normal at Σ_1 reverses the sign of the tangential boundary operator and so changes the associated positive spectral projection to a negative spectral projection.

Altogether then, we see that the range of (11.54) is a compact perturbation of the range of

$$\operatorname{im}\Pi^+ \oplus \operatorname{im}\Pi^- : \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1} \circlearrowleft .$$

This is convenient because $\operatorname{im}\widetilde{P}_0^+$ certainly not a direct sum of a subspace of $\widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0}$ with a subspace of $\widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1}$, but the above analysis tells us it is a compact perturbation of this. From this, we easily deduce

Lemma 11.4 *Suppose we have*

(F0) $B^{k+1/2+s,2}\widetilde{L_0}$ is Fredholm with $\operatorname{im}\Pi^+$ in $\widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0}$;

(F1) $B^{k+1/2+s,2}L_1$ is Fredholm with $\operatorname{im}\Pi^-$ in $\widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1}$.

Then $B^{k+1/2+s,2}(\widetilde{L_0} \oplus L_1)$ Fredholm with $r(\ker\widetilde{\mathcal{H}}_0)$ in $\widetilde{\mathcal{T}}^{k+1/2+s}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}^{k+1/2+s}_{\Sigma_1}$, where $\widetilde{\mathcal{H}}_0$ is the operator in (11.52). This implies (11.52) is Fredholm.

If $\mathfrak{L}$ is a monopole Lagrangian, Theorem 11.7 tells us that (F0) is satisfied for L_0 a tangent space to $\mathfrak{L}$. Thus, (F1) is the only condition that needs to be satisfied. This latter condition is a generic open condition, and so we see that there is great freedom in our choice of L_1. Moreover, if we let $\chi : [0,1] \to \mathbb{R}^+$ be a smooth cutoff function, $\chi(v) = 1$ on $v \le 1/2$ and $\chi(v) = 0$ for $v \ge 3/4$, then for any $(b, \psi, \xi) \in \widetilde{\mathcal{T}}^{(k+1,s),2}$, we have

$$\chi(b, \psi, \xi) \in \widetilde{\mathcal{T}}^{(k+1,s),2}_{[0,1]\times\Sigma, L_0, L_1}$$

for any choice of L_1. Furthermore, it is these truncated configurations[16] for which we will bootstrapping regularity in the proof of Theorem A. Thus, we see that in this sense, the choice of L_1 is just a "dummy" boundary condition to make the operator (11.52) Fredholm.

Assuming (F0), we have described a sufficient condition (F1) that makes (11.52) Fredholm. We are interested of course in the analog of Theorem 11.2 on anisotropic spaces, which means we need to now promote everything to time-varying domains and study the operator $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ on anisotropic function spaces. We are thus led to consider the space

$$\mathrm{Maps}^{(k,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}), \tag{11.55}$$

the closure of the space of smooth paths $\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma})$ in the topology $B^{(k,s),2}((S^1 \times [0,1]) \times \Sigma)$. Thus, the s measures anisotropy in the Σ directions only. Likewise, we can define $\mathrm{Maps}^{(k,s)}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma, L(t), L_1})$ as the anisotropic analogue of (11.31), namely,

$$\mathrm{Maps}^{(k,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma, L(t), L_1}) := B^{k,2}(S^1, \widetilde{\mathcal{T}}^{(0,s),2}_{[0,1]\times\Sigma}) \cap L^2(S^1, \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma, L(t), L_1}). \tag{11.56}$$

The notation involved, while systematic, is unfortunately a nightmare. The below diagram summarizes all the spaces involved with their appropriate topologies:

[16] More properly, the truncations of the time-dependent configurations on $S^1 \times [0,1] \times \Sigma$.

$$
\begin{array}{ccccc}
\widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}^{k+1/2+s,2}_{\Sigma_1} & \xleftarrow{r=(r_0,r_1)} & \widetilde{\mathcal{T}}^{(k+1,s),2}_{[0,1]\times\Sigma} & \xrightarrow{\widetilde{\mathcal{H}}_0} & \\
\uparrow & & \uparrow & & \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma} \\
B^{(k+1/2+s),2}(\widetilde{L} \oplus L_1) & \xleftarrow{r} & \widetilde{\mathcal{T}}^{(k+1,s),2}_{[0,1]\times\Sigma,L,L_1} & \xrightarrow{\widetilde{\mathcal{H}}_0} &
\end{array}
$$

$$
\begin{array}{ccccc}
\mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{\mathcal{T}}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}_{\Sigma_1}) & \xleftarrow{r} & \mathrm{Maps}^{(k+1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) & \xrightarrow{\frac{d}{dt}+\widetilde{\mathcal{H}}_0} & \\
\uparrow & & \uparrow & & \mathrm{Maps}^{(k,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) \\
\mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{L}(t) \oplus L_1) & \xleftarrow{r} & \mathrm{Maps}^{(k+1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L(t),L_1}) & \xrightarrow{\frac{d}{dt}+\widetilde{\mathcal{H}}_0} &
\end{array}
\tag{11.57}
$$

We want to study the operator

$$
\frac{d}{dt} + \widetilde{\mathcal{H}}_0 : \mathrm{Maps}^{(k+1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L(t),L_1}) \to \mathrm{Maps}^{(k,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) \tag{11.58}
$$

where $L(t)$ is a family of tangent spaces to $\mathfrak{L}$ along a smooth path, corresponding to the boundary value problem we are trying to study. A priori, it is not at all obvious why this operator should be Fredholm and satisfy a corresponding estimate as in Theorem 11.2. Indeed, we can no longer use simple self-adjointness techniques, since the Hilbert space $\widetilde{\mathcal{T}}^{(0,s),2}_{[0,1]\times\Sigma}$ no longer admits $\widetilde{\mathcal{H}}_0$ as an (unbounded) symmetric operator when $s > 0$. (Indeed, the inner product on $\widetilde{\mathcal{T}}^{(0,s),2}_{[0,1]\times\Sigma}$ is no longer defined in terms of the bundle metrics implicit in the definition of $\widetilde{\mathcal{T}}$ but contains operators in the Σ direction which capture the anisotropy.) This means we no longer have the integration by parts formula (11.38), nor can we apply the spectral theorem as in (11.44) to understand the resolvent of $\widetilde{\mathcal{H}}_0$ on anisotropic function spaces. However, all is not lost, since we can still prove an analogous estimate to (11.44) in the anisotropic setting. This is because the resolvent of $\widetilde{\mathcal{H}}_0$, which is a resolvent associated to an elliptic boundary value problem, is a pseudodifferential type operator, and such an operator lends itself to estimates on a variety of function spaces, including anisotropic spaces. Indeed, this is the reason we proved Theorem 11.2 using resolvents as an alternative method to the integration by parts method, since the robust methods of pseudodifferential operator theory will carry over to anisotropic spaces.

On anisotropic spaces, the resolvent we wish to understand is the resolvent of

$$
\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{(1,s),2}_{[0,1]\times\Sigma,L_0,L_1} \to \widetilde{\mathcal{T}}^{(0,s),2}_{[0,1]\times\Sigma}, \tag{11.59}
$$

and we want the estimate

$$\|(i\tau - \widetilde{\mathcal{H}}_0)^{-1}\|_{B^{(0,s),2}([0,1]\times\Sigma)} \leq O(\tau^{-1}). \tag{11.60}$$

As in the proof of Theorem 11.2, we assume here that (11.59) is invertible and self-adjoint for $s = 0$, which we can always do by perturbing $\widetilde{\mathcal{H}}_0$ by a bounded operator. By doing so, the resolvent in (11.60) makes sense for all $\tau \in \mathbb{R}$.

There is a well-developed theory for understanding the resolvent of elliptic boundary value problems, dating back to the work of Seeley in [47]. There the boundary conditions considered were differential and later extensions were made to pseudodifferential boundary conditions satisfying certain hypotheses (see e.g. [18] and [17]). For us, the tangent space to a monopole Lagrangian is only "nearly" pseudodifferential, in the sense that its tangent spaces are given by the range of projections which differ from a pseudodifferential operator by a smoothing operator (see Theorem 3.13(ii)). However, after a detailed analysis, one can adapt the methods of [18] and [17] to carry over to our present situation. In carrying out this analysis, we should remark here that it is key that the operator (11.59 is self-adjoint for $s = 0$. We develop a sufficiently general framework for the construction of the resolvent of an elliptic boundary value problem in Part IV, and via Theorem 15.32 and Corollary 15.34, we prove the resolvent estimates we need on anisotropic function spaces.

Having made the above remarks, let us finally state the generalization of Theorem 11.2 to the anisotropic situation. In order to do this, we have to introduce the anisotropic version of Definition 11.1, so that the maps which straighten the domains are well-behaved on anisotropic spaces. For this purpose, define the subspace

$$\widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L} = \{x \in \widetilde{\mathcal{T}}_{[0,1]\times\Sigma},\ r_0(x) \in \widetilde{L}\} \subset \widetilde{\mathcal{T}}_{[0,1]\times\Sigma},$$

where we only impose boundary conditions on $\Sigma = \Sigma_0$.

Definition 11.5 Let $L(t)$ be a smoothly varying family of subspaces[17] of $\mathcal{T}_\Sigma$, $t \in S^1$ or $\mathbb{R}$. We say that the $L(t)$ are *anisotropic regular* if for every $t_0 \in \mathbb{R}$, there exists an open interval $I \ni t_0$ such that for every $t \in I$, there exist isomorphisms $S(t) : \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}$ satisfying the following properties for all nonnegative integers k and every $s \in [0,1]$:

(i) The map $S(t)$ extends to an isomorphism $S(t) : \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma}$. Furthermore, $S(t)$ acts as the identity on $[1/2,1]\times\Sigma$, i.e., for every $(b,\phi,\xi) \in \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma}$, we have

$$S(t)\big((b,\phi,\xi)\big)\Big|_{[1/2,1]\times\Sigma} = (b,\phi,\xi)\Big|_{[1/2,1]\times\Sigma}.$$

(ii) The map $S(t)$ straightens the family $L(t)$ in the sense that $S(t) : \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma,L(t_0)} \to \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma,L(t)}$ is an isomorphism ($k \geq 1$).

(iii) The commutator of $[D, S(t)]$ where $D : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ is any first order differential operator, is an operator bounded on $\widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma}$.

[17] See Definition 18.9.

Observe that if the family $L(t)$ is anisotropic regular then it is also regular, since we can take $s = 0$ in the above and extend $S(t)$ by the identity to the rest of Y.

Theorem 11.6 *Assume the family of spaces $L(t) \subset \mathcal{T}_{\Sigma_0}$, $t \in S^1$, and the space $L_1 \subseteq \widetilde{\mathcal{T}}_{\Sigma_1}$ satisfy the following:*

(i) The $L(t)$ are anisotropic regular.

(ii) The operators

$$\widetilde{\mathcal{H}}_0 : \widetilde{\mathcal{T}}^{(k+1,s),2}_{[0,1]\times\Sigma,L_0,L_1} \to \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma}, \qquad t \in S^1, \tag{11.61}$$

are Fredholm for all $k \geq 0$ and $s \in [0,1]$, and moreover, for $k = s = 0$, are self-adjoint.

(iii) The resolvent estimate (11.60) holds for all $s \in [0,1]$.

Then

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 : \mathrm{Maps}^{(k+1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L(t),L_1}) \to \mathrm{Maps}^{(k,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) \tag{11.62}$$

is Fredholm for all $k \geq 0$ and $s \in [0,1]$, and we have the elliptic estimate

$$\|x\|_{B^{(k+1,s),2}((S^1\times[0,1])\times\Sigma)} \leq C\left(\left\|\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_0\right)x\right\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)} + \|x\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)}\right). \tag{11.63}$$

Furthermore, if $x \in \mathrm{Maps}^{(1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L(t),L_1})$ satisfies $(\frac{d}{dt}+\widetilde{\mathcal{H}}_0)x \in \mathrm{Maps}^{(k,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma})$, then $x \in \mathrm{Maps}^{(k+1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L(t),L_1})$ and it satisfies (11.63).

Proof The same proof of (11.45) using the Fourier transform shows that (11.60) implies

$$\|\partial_t x\|_{B^{(0,s),2}((S^1\times[0,1])\times\Sigma)} \leq C\|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{B^{(0,s),2}((S^1\times[0,1])\times\Sigma)}, \tag{11.64}$$

Here, we used that $B^{s,2}([0,1]\times\Sigma)$ is a Hilbert space, so that we have a Plancherel theorem on $B^{(0,s),2}((S^1\times[0,1])\times\Sigma) = L^2(S^1, B^{s,2}([0,1]\times\Sigma))$. From (11.64), the same reasoning we used to derive (11.46) shows that we now the anisotropic estimate

$$\|x\|_{B^{(1,s),2}((S^1\times[0,1])\times\Sigma)} \leq C(\|(\partial_t + \widetilde{\mathcal{H}}_0)x\|_{B^{(0,s),2}((S^1\times[0,1])\times\Sigma)} + \|x\|_{B^{(1,s),2}((S^1\times[0,1])\times\Sigma)}). \tag{11.65}$$

This proves (11.63) for $k = 0$. From here, commuting time derivatives as in the proof of (11.36) shows that (11.63) holds for all $k \geq 0$. For the last statement, see Remark 11.3. □

Summarizing, we have studied the operator $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ on the manifold with boundary $S^1 \times Y$, both on the usual L^2 spaces and on anisotropic spaces. We have listed general properties that families of subspaces $L(t)$, serving as boundary conditions for $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$, should satisfy if the spaces

$$\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{L(t)}), \qquad \mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma,L(t),L_1}), \tag{11.66}$$

in suitable function space completions, are to yield a domain for which the operator $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ is Fredholm. We have phrased matters in this generality, because this is the general model for the linearized Seiberg-Witten equations with Lagrangian boundary conditions, where the $L(t)$ come from linearizing a path along the Lagrangian $\mathfrak{L}$. This allows us to understand the general framework for these equations, in particular, which Lagrangians are suitable for a well-posed boundary value problem. We make further remarks on this and related issues in Section 5.

Of course, all of this discussion would be fruitless if we could not produce any examples of Lagrangians which satisfy the properties we have imposed. Fortunately, for $\mathfrak{L}$ a monopole Lagrangian, all the properties we have used in Theorem 11.6 hold. We have the following theorem.

Theorem 11.7 *Let $\mathfrak{L}$ be a monopole Lagrangian and let $\gamma \in \mathrm{Maps}(S^1, \mathfrak{L})$ a smooth path. Define $L(t) = T_{\gamma(t)}\mathfrak{L}$, $t \in S^1$. Then there exists $L_1 \subset \widetilde{\mathcal{T}}_{\Sigma_1}$ such that all the hypotheses (i)-(iii) of Theorem 11.6 hold.*

Proof (i) Since $\mathfrak{L} \subset \mathfrak{C}(\Sigma)$ is a submanifold, then the tangent spaces to any smooth path $\gamma \in \mathrm{Maps}(S^1, \mathfrak{L})$ automatically form a smoothly varying family of subpaces of $\mathcal{T}_\Sigma$. By Theorem 3.13(ii), we know that each tangent space $L(t) = T_{\gamma(t)}\mathfrak{L}$ is the range of a "Calderon projection"

$$P^+(t) := P^+_{\tilde{\gamma}(t)} : \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma. \tag{11.67}$$

Here $\tilde{\gamma} \in \mathrm{Maps}(I, \mathcal{M})$ is a smooth path that lifts $\gamma \in \mathrm{Maps}(I, \mathcal{L})$, i.e. $r_\Sigma(\tilde{\gamma}(t)) = \gamma(t)$ for all t. That such a smooth lift exists follows from the techniques used in the proof of Theorem 10.9(iv). For each t, the resulting projection $P^+(t)$ extends to a bounded map, in particular, on $\mathcal{T}^{s,2}_\Sigma$ for all $s \geq 0$. Furthermore, it differs from a pseudodifferential projection π^+ by an operator $T(t) := P^+(t) - \pi^+$ that smooths by one derivative, i.e., $T(t) : T^{s,2}_\Sigma \to T^{s+1,2}_\Sigma$ for all $s \geq 0$. Indeed, because $\tilde{\gamma}$ is smooth, one can check from the arithmetic of Theorem 3.13 that the maps $P^+(t)$ and $T(t)$ have the mapping properties on all the function spaces just stated.

To construct the straightening maps $S(t)$ in Definition 11.5, we construct straightening maps on the boundary using Lemma 18.10, and then extend these to maps in a collar neighborhood of the boundary in a slicewise fashion. In detail, given any $t_0 \in S^1$, say $t_0 = 0$, Theorem 18.10 tells us that there exists a time interval $I \ni 0$ such that we have straightening maps

$$S_\Sigma(t) : \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma, \qquad t \in I,$$

with each $S_\Sigma(t)$ is an isomorphism and $S_\Sigma(t)(L(0)) = L(t)$. Moreover, since the closures $B^{s,2}L(t)$ are also smoothly varying and complemented in $\mathcal{T}^{s,2}_\Sigma$ for all $s \geq 0$ (again by the Theorem 3.13), the straightening maps extend to the Besov closures as well, and we have

$$S_\Sigma(t) : \mathcal{T}^{s,2}_\Sigma \to \mathcal{T}^{s,2}_\Sigma, \qquad t \in I,\ s \geq 0, \tag{11.68}$$

with $S_\Sigma(t)(B^{s,2}L(0)) = B^{s,2}L(t)$.

We now use these boundary straightening maps to construct straightening maps on $\widetilde{\mathcal{T}}_{[0,1]\times\Sigma}$. Let $(b, \phi, \xi) \in \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}$. In the collar neighborhood $[0,1] \times \Sigma$, let $v \in [0,1]$ be the inward normal coordinate and write $b = b_1 + \beta dv$ in terms of its tangential and normal components, respectively, where $b_1 \in \Gamma([0,1], \Omega^1(\Sigma; i\mathbb{R}))$ and $\beta \in \Gamma([0,1], \Omega^0(\Sigma; i\mathbb{R}))$. We will use the $S_\Sigma(t)$ maps to straighten out the tangential components $(b_1, \phi)|_\Sigma$ and we need not do anything to the β_1 and ξ. More precisely, let $h = h(v)$ be a smooth cutoff function, $0 \leq h(v) \leq 1$, where $h(v) = 1$ for $v \leq 1/4$ and $h = 0$ for $v \geq 1/2$. We define

$$\begin{aligned} S(t) : \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} &\to \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} \\ (b, \phi, \xi) &\mapsto h(v)S_\Sigma(t)(b_1(v), \phi(v)) + (1 - h(v))(b_1(v), \phi(v)) + \beta dv + \xi, , \end{aligned} \tag{11.69}$$

where $S(t)$ is defined to be the identity outside of $[0,1] \times \Sigma$. Thus, at $v = 0$, $S(t)$ acting on the tangential components $(b_1(0), \phi(0))$ is just the map $S_\Sigma(t)$, on $v \geq 1/2$, the map $S(t)$ is the identity, and in between, we linearly interpolate. For all v, we do nothing to βdv and ξ. We now have to check that all the properties of Definition 11.5 hold. For (i), we first have that (11.69) is an isomorphism in the smooth setting for all $t \in I$, where I is sufficiently small. Indeed, at $t = 0$, the map $S_\Sigma(0)$ is just the identity and hence so is $S(0)$. For small enough t, the map $S_\Sigma(t)$ is sufficiently close to the identity that all its linear interpolants with the identity map on $\mathcal{T}_\Sigma$ are still isomorphisms. Shrinking I if necessary, it follows that (11.69) is an isomorphism for all $t \in I$. It remains to show that $S(t)$ is bounded on anisotropic Besov spaces. However, this follows from a similar analysis as was done in Lemma 10.6, since although $S(t)$ is not time-independent, it is smoothly so. Indeed, the mapping properties of $S(t)$ are determined from $S_\Sigma(t)$ acting slicewise in the v direction. This latter slicewise map clearly acts on integer Sobolev spaces because of the Leibnitz rule, the smoothness of $S_\Sigma(t)$, and (11.68). The Fubini property and interpolation property of Besov spaces, as explained in Lemma 10.6, now show that the v-slicewise $S_\Sigma(t)$ is bounded on anisotropic Besov spaces and hence so is $S(t)$. This proves (i) in Definition 11.5. Next, for (ii), it follows from $S_\Sigma(t)(B^{k+s-1/2,2}L(0)) = B^{k+s-1/2,2}L(t))$ and the boundedness of $S(t)$ on $\widetilde{\mathcal{T}}^{(k,s),2}_{[0,1],\Sigma}$ that

$$S(t) : \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma,L(0)} \to \widetilde{\mathcal{T}}^{(k,s),2}_{[0,1]\times\Sigma,L(t)}, \qquad k \geq 1. \tag{11.70}$$

Finally, for the commutator property (iii) in Definition 11.5, we need only work in the collar neighborhood $[0,1] \times \Sigma$, since $S(t)$ is the identity outside of it. There are two cases. If D is a differential operator in the Σ directions, then $[D, S(t)]$ is essentially given by $[D, P^+(t)]$ and $[D, T(t)]$, both of which yield bounded operators; the former because the commutator of a first and zeroth order pseudodifferential operator is a zeroth order pseudodifferential operator (hence bounded) and the latter because $T(t)$ is already smoothing by one derivative in the Σ directions. If D is a differential operator in the v direction, then $[S(t), D]$ is still a bounded operator, since we just differentiate the smooth cutoff function $h(v)$ in the commutator. Altogether, this proves $S(t)$ satisfies all properties of Definition 11.5 and hence, the family $L(t)$ is anisotropic regular.

(ii) Using Lemma 11.4, if we choose any L_1 such that (F1) is satisfied, then it suffices to prove that condition (F0) holds for $L_0 = L(t)$ for every t. It is here where we use that the monopole Lagrangian $\mathfrak{L} = \mathcal{L}(Y')$ comes from a manifold Y' such that $\partial Y' = -\partial Y$. Following the discussion in Section 3.3 of Part I, on Y, the operator $\widetilde{\mathcal{H}}_0$ is a Dirac operator that decomposes as the sum of two Dirac operators,

$$\widetilde{\mathcal{H}}_0 = D_{\mathrm{dgc}} \oplus D_{B_{\mathrm{ref}}}, \tag{11.71}$$

the div-grad-curl operator

$$D_{\mathrm{dgc}} = \begin{pmatrix} *d & -d \\ d^* & 0 \end{pmatrix} : \Omega^1(Y; i\mathbb{R}) \oplus \Omega^0(Y; i\mathbb{R}) \circlearrowleft,$$

on differential forms, and the Dirac operator

$$D_{B_{\mathrm{ref}}} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$$

on spinors. Thus, the tangential boundary operator $\mathsf{B} : \widetilde{\mathcal{T}}_\Sigma \to \widetilde{\mathcal{T}}_\Sigma$ associated to $\widetilde{\mathcal{H}}_0$ on $\Sigma = \Sigma_0$ splits as a sum

$$\mathsf{B} = \mathsf{B}_{\mathrm{dgc}} \oplus \mathsf{B}_{\mathcal{S}} \tag{11.72}$$

of the tangential boundary operators associated to D_{dgc} and $\mathcal{D}_{B_{\mathrm{ref}}}$, respectively. We get an associated spectral decomposition of $\widetilde{\mathcal{T}}_\Sigma$ via

$$\widetilde{\mathcal{T}}_\Sigma = \mathcal{Z}^+ \oplus \mathcal{Z}^- \oplus \mathcal{Z}^0 \tag{11.73}$$

given by the positive, negative, and zero eigenspace decomposition of B. Furthermore, by (11.72), we have

$$\mathcal{Z}^\pm = (\mathcal{Z}_e^\pm \oplus \mathcal{Z}_c^\pm) \oplus \mathcal{Z}_{\mathcal{S}}^\pm \tag{11.74}$$

where

$$\mathcal{Z}_e^\pm \oplus \mathcal{Z}_c^\pm \subset \Omega^1(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R})$$

and

$$\mathcal{Z}_{\mathcal{S}}^\pm \subseteq \Gamma(\mathcal{S}_\Sigma)$$

are the positive and negative eigenspaces associated to $\mathsf{B}_{\mathrm{dgc}}$ and $\mathsf{B}_{\mathcal{S}}$, respectively, see Lemma 3.8 and (3.77).

Thus, to show (F0) in Lemma 11.4, we have to show that $B^{k+s+1/2,2}\widetilde{L(t)}$ is Fredholm with $B^{k+s+1/2,2}\mathcal{Z}^+$. By Theorem 3.13(i), we have that $B^{k+s+1/2,2}L(t)$ is a compact perturbation of $B^{k+s+1/2,2}(\operatorname{im} d \oplus \mathcal{Z}_{\mathcal{S}}^-)$ in $\mathcal{T}_\Sigma^{k+s+1/2,2}$. Note the important minus sign in the last factor. This minus sign arises because when we apply Theorem 3.13, we apply it to the manifold Y', and since we have the opposite orientation $\partial Y' = -\partial Y$, the tangential boundary operators for the operators on Y' differ by a minus sign from the corresponding ones on Y. Altogether then, $B^{k+s+1/2,2}\widetilde{L(t)}$ is a compact perturbation of

$$B^{k+s+1/2,2}(\operatorname{im} d \oplus \mathcal{Z}_{\mathcal{S}}^- \oplus 0 \oplus \Omega^0(\Sigma; i\mathbb{R})) \subset \widetilde{\mathcal{T}}_\Sigma^{k+s+1/2,2}. \tag{11.75}$$

From the definition of $\mathcal{Z}_e^+$ and $\mathcal{Z}_c^+$ in Lemma 3.8, one can now easily see that (11.75) is Fredholm with $B^{k+s+1/2,2}\mathcal{Z}^+$ via (11.74). Thus, (11.75) and hence $B^{k+s+1/2,2}\widetilde{L(t)}$ is Fredholm with $B^{k+s+1/2,2}\mathcal{Z}^+$. So (F0) is satisfied, and this proves the Fredholm property of (11.61) by Lemma 11.4.

For $k = s = 0$, the operator (11.61) is symmetric since $B^{1/2,2}\widetilde{L(t)} \subseteq \widetilde{\mathcal{T}}_\Sigma^{1/2,2}$ is an isotropic subspace. Here, we suppose the subspace L_1 chosen in (ii), which satisfies (F1), is such that $B^{1/2,2}L_1 \subset \widetilde{\mathcal{T}}_{\Sigma_1}^{1/2,2}$ is a Lagrangian subspace. From this, it turns out that since $B^{1/2,2}\widetilde{L(t)} \subseteq \widetilde{\mathcal{T}}_\Sigma^{1/2,2}$ is a Lagrangian subspace which furthermore satisfies (F0), then

(11.61) is self-adjoint[18]. This follows from the general and abstract framework of finding self-adjoint extensions of closed-symmetric operators by finding Lagrangian subspaces of a suitable quotient Hilbert space, dating back to classical work of von-Neumann. Here, the relevant theorem is Theorem 22.4. This shows (11.61) is self-adjoint for $k = 0$.

(iii) This follows from Theorem 3.13(ii) and Corollary 15.34. □

Thus, both Theorems 11.2 and 11.6 hold for the linearized operator arising from the (gauge-fixed) Seiberg-Witten equations with Lagrangian boundary condition determined by a monopole Lagrangian. In particular, from Theorem 11.2, we see that the linearized operator of our boundary value problem is a Fredholm operator. This proves

Theorem 11.8 *(Fredholm Property) Let $\mathfrak{L}$ be a monopole Lagrangian. Consider the equations (9.7) on $S^1 \times Y$, where we impose the gauge fixing condition*

$$d^*(A - A_0) = 0, \qquad (A - A_0)|_{S^1 \times \Sigma} = 0, \tag{11.76}$$

where A_0 is a smooth connection. Then the linearization of the system (9.7) and (11.76) at a smooth configuration $(A, \Phi) = (B(t) + \alpha(t)dt, \Phi(t))$ determines an operator

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 - D_{(A,\Phi)}N_{(A_0,\Phi_0)} : \mathrm{Maps}^{k+1,2}(S^1, \widetilde{\mathcal{T}}_{L(t)}) \to \mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{L(t)}), \tag{11.77}$$

where $\widetilde{\mathcal{H}}_0$ and $N_{(A_0,\Phi_0)}$ are given by (11.12) and (11.15), respectively, $L(t) = T_{r_\Sigma(B(t),\Phi(t))}\mathcal{L}$, and where $k \geq 0$. The operator (11.77) is a Fredholm operator for all $k \geq 0$.

In particular, this means that if we have transversality for our Seiberg-Witten system (say by perturbing the equations in the interior in a mild way), then the moduli space of solutions modulo gauge to our boundary value problem is finite dimensional.

We have one more corollary to the above analysis which we will need in the next section, which yields for us the inhomogeneous version of Theorem 11.6.

Corollary 11.9 *Let $\gamma \in \mathrm{Maps}(S^1, \mathfrak{L})$ be a smooth path, let $L(t) = T_{\gamma(t)}\mathfrak{L}$, and let L_1 be as in Theorem 11.7. Then we have the following:*

(i) The space

$$\mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{L(t)} \oplus L_1) := \{z \in \mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{\mathcal{T}}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}_{\Sigma_1}) : z(t) \in \widetilde{L(t)} \oplus L_1, \text{ for all } t \in S^1\} \tag{11.78}$$

is Fredholm with $r\left(\ker\left(\partial_t + \widetilde{\mathcal{H}}_0\right)\right) \subset \mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{\mathcal{T}}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}_{\Sigma_1})$, for all $k \geq 0$ and $0 \leq s \leq 1$.

[18] Not every Lagrangian subspace of $\widetilde{\mathcal{T}}^{1/2,2}_\Sigma \oplus \widetilde{\mathcal{T}}^{1/2,2}_{\Sigma_1}$ will yield a self-adjoint operator. For instance, $r(\ker \widetilde{\mathcal{H}}_0) \subset \widetilde{\mathcal{T}}^{1/2,2}_\Sigma \oplus \widetilde{\mathcal{T}}^{1/2,2}_{\Sigma_1}$ is a Lagrangian subspace, but this boundary condition is not a self-adjoint boundary condition for $\widetilde{\mathcal{H}}_0$.

(ii) *The intersection of (11.78) and* $r\left(\ker\left(\partial_t + \widetilde{\mathcal{H}}_0\right)\right)$ *is spanned by finitely many smooth elements, and the span of these two spaces is complemented by a space spanned by finitely many smooth elements.*

(iii) *There exists a projection*

$$\Pi : \mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{\mathcal{T}}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}_{\Sigma_1}) \circlearrowleft \tag{11.79}$$

such that $\ker \Pi$ *is (11.78) and* $\operatorname{im}\Pi$ *is of finite codimension in* $r\left(\ker\left(\partial_t + \widetilde{\mathcal{H}}_0\right)\right)$. *The map* Π *is independent of* $k \geq 0$ *and* $s \in [0,1]$. *Moreover, we have the inhomogeneous elliptic estimate*

$$\|x\|_{B^{(k+1,s),2}((S^1\times[0,1])\times\Sigma)} \leq C\Bigg(\left\|\left(\partial_t + \widetilde{\mathcal{H}}_0\right)x\right\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)} + \\ \|\Pi r(x)\|_{B^{(k+1/2,s),2}(S^1\times(\Sigma_0\cup\Sigma_1))} + \|x\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)}\Bigg). \tag{11.80}$$

Proof (i) The space (11.78) is precisely the space of boundary values of the domain of (11.62). Since the operator (11.62) is Fredholm, (11.78) is Fredholm with $r\left(\ker\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_0\right)\right)$, the boundary values of the kernel of

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 : \mathrm{Maps}^{(k+1,s),2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) \to \mathrm{Maps}^{k,2}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma})$$

where no boundary conditions are imposed. This follows from Theorem 15.23.

(ii) The intersection of the two spaces consists of smooth elements because we have the elliptic estimate (11.63) for all k, which tells us that all elements in the kernel of (11.62) are smooth. The same analysis applies to the adjoint problem, and so the cokernel of (11.62) (that is, the orthogonal complement of its range) is also spanned by smooth configurations. We now apply Theorem 15.23.

(iii) This follows from Theorem 15.25. By (ii), the projection Π differs from the projection $(1 - \Pi_{\mathcal{U}})$ in (15.39) by a smooth error (where $\mathcal{U}$ is taken to be (11.78)), and so (11.80) follows from (15.39). The map Π is independent of k and s, since the kernel and cokernel of 11.62 are independent of k and s and the Fredholm property in (i) holds for every k and s. □

12 Proofs of the Main Theorems

In the previous section, we studied the linearized Seiberg-Witten equations with abstract Lagrangian boundary conditions. In the course of doing so, we found that Lagrangians satisfying certain analytic properties yield elliptic estimates for the linearized Seiberg-Witten equations. Furthermore, we showed that monopole Lagrangians satisfy all such properties. Thus, with $\mathfrak{L}$ a monopole Lagrangian, we can now prove our main theorems, using both the linear analysis in the previous section, and the nonlinear analysis in Section 2 concerning the space of paths through $\mathfrak{L}$. It is convenient to prove the results on $S^1 \times Y$. We then explain how a standard patching argument proves the result on $\mathbb{R} \times Y$.

Recall that when $p = 2$, the Besov spaces $B^{s,2}$ are the usual fractional Sobolev spaces $H^s = H^{s,2}$, for all $s \in \mathbb{R}$. When s is a nonnegative integer, the spaces $B^{s,2} = H^{s,2}$ are also denoted by $W^{s,2}$. We will use either notation $B^{s,2}$ or $H^{s,2}$ wherever convenient.

Theorem 12.1 *Let $p > 4$, and let $A = B(t) + \alpha(t)dt \in H^{1,p}\mathcal{A}(S^1 \times Y)$ and $\Phi \in H^{1,p}\Gamma(\mathcal{S}^+)$ solve the boundary value problem*

$$SW_4(A, \Phi) = 0, \tag{12.1}$$

$$r_\Sigma(B(t), \Phi(t)) \in \mathfrak{L}^{1-2/p,p}, \quad \forall t \in S^1, \tag{12.2}$$

where $\mathfrak{L}$ is a monopole Lagrangian. Then there exists a gauge transformation $g \in H^{2,p}\mathcal{G}_{\mathrm{id}}(S^1 \times Y)$ such that $g^*(A, \Phi)$ is smooth. In particular, if A is in Coulomb-Neumann gauge with respect to any smooth connection, i.e. (11.76) holds, then (A, Φ) is smooth.

Proof From the linear analysis in the previous section, we know we can find a gauge in which the equations are a semilinear elliptic equation (in the interior), with quadratic nonlinearity, as in (11.13). Since $H^{k,p}(S^1 \times Y)$ is an algebra for $k \geq 1$ and $p > 4$, it follows that we can elliptic bootstrap in the interior to any desired regularity. Thus, we need only prove regularity near the boundary.

From here, the proof proceeds in four main steps. The first step is to rewrite the equations in a suitable gauge so that the linear portion of the equations satisfy all the hypotheses the previous section. In particular, the linearized equations now satisfy elliptic estimates on anisotropic function spaces, where the anisotropy is in the Σ direction. From this, the second step is to gain regularity for (A, Φ) in the Σ directions in a neighborhood of the boundary. Here, we use the results from Section 2, namely Theorems 10.9 and 10.10, that the nonlinear part of the chart maps for the space of paths through $\mathfrak{L}$ smooth in the Σ directions. Moreover, it is here that the complicated choice of topologies appearing in these theorems, particularly in Theorem 10.10, will serve their purpose. Using the anisotropic linear theory of Section 3, specifically Corollary 11.9, we then gain regularity for (A, Φ) in the Σ directions, where the linear theory can be applied because the nonlinear contribution from the boundary condition is smoothing in the Σ directions. The third step is to gain regularity in the time direction and normal direction to Σ using the theory of Banach space valued Cauchy-Riemann equations due to Wehrheim [52] which we adapt to our needs in Section 16 of Part IV. Once we have gained some regularity in all the directions, then in our final step, we bootstrap to gain regularity to any desired order.

Step One: In the previous section, we found a suitable gauge in which the Seiberg-Witten equations become a semilinear elliptic equation with quadratic nonlinearity, namely, we obtained the system (11.13) for the equations in the interior. We wish to do the same here, only now (A, Φ) is not smooth. Furthermore, we must choose the base configuration (A_0, Φ_0) about which we linearize our configuration (A, Φ) carefully.

So choose $(A_0, \Phi_0) \in \mathfrak{C}(S^1 \times Y)$ a smooth configuration close to (A, Φ), where we will define this more precisely in a moment. In the usual way, write $A_0 = B_0(t) + \alpha_0(t)dt$ as a path of connections $B_0(t)$ on Y plus its temporal part $\alpha_0(t)$, and write $\Phi_0 = \Phi_0(t)$ as a path of spinors on Y. Then we can find a gauge transformation $g \in H^{2,p}\mathcal{G}_{\mathrm{id}}(S^1 \times Y)$, that places A in Coulomb-Neumann gauge with respect to A_0, i.e.,

$$d^*(g^*A - A_0) = 0, \qquad *(g^*A - A_0)|_{S^1\times\Sigma} = 0. \tag{12.3}$$

This g is determined by writing $g = e^f$, where $f \in \Omega^0(S^1 \times Y; i\mathbb{R})$, and solving the inhomogeneous Neumann problem

$$\Delta f = d^*(A - A_0) \tag{12.4}$$

$$*df|_{S^1\times\Sigma} = *(A - A_0)|_{S^1\times\Sigma} \tag{12.5}$$

This equation has a unique solution $f \in H^{2,p}\Omega^0(Y; i\mathbb{R})$, up to constants, by the standard elliptic theory of the Neumann Laplacian.

Redefine (A, Φ) by the gauge transformation so obtained above, so that we have

$$d^*(A - A_0) = 0, \qquad *(A - A_0)|_{S^1\times\Sigma} = 0. \tag{12.6}$$

We want to gain regularity for the difference $(A, \Phi) - (A_0, \Phi_0)$, which we can write as the triple

$$(b, \phi, \xi) \in H^{1,p}\mathrm{Maps}(S^1, \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S}) \oplus \Omega^0(Y; i\mathbb{R})),$$

where $b = b(t)$ is $B(t) - B_0(t)$, $\phi = \phi(t)$ is $\Phi(t) - \Phi_0(t)$, and $\xi(t) = \alpha(t) - \alpha_0(t)$. From now on, we just write

$$(b, \phi, \xi) = (A, \Phi) - (A_0, \Phi_0). \tag{12.7}$$

for short.

Our goal is to show that (b, ϕ, ξ) is smooth. As shown in Section 3, the configuration (b, ϕ, ξ) satisfies (11.13) and (11.14). We now have to add in the nonlinear Lagrangian boundary condition (12.2) to these equations. To express this in terms of (b, ϕ, ξ) requires that we chose (A_0, Φ_0) sufficiently close to (A, Φ), as we now explain. Recall from Theorem 10.9 that for any path $\gamma \in \mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L})$, there is a local chart map $\mathcal{E}_\gamma$ which maps a neighborhood of 0 in the tangent space $T_\gamma\mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L})$ diffeomorphically onto a neighborhood of $\gamma \in \mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L})$. Furthermore, by construction, the chart maps contain a $C^0(S^1, B^{s',p}(\Sigma))$ neighborhood of $\mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L})$, for any $1/2 < s' \leq 1 - 2/p$, and the size of this neighborhood can be chosen uniformly on small $C^0(S^1, B^{s',p}(\Sigma))$ neighborhoods of γ. It follows that if the (smooth) (A_0, Φ_0) is sufficiently $H^{s,p}(S^1 \times Y)$ close to (A, Φ), with $s > 1/2 + 2/p$, then on the boundary, the associated smooth path

$$\gamma_0 := \widehat{r_\Sigma}(B_0(t), \Phi_0(t)) \in \mathrm{Maps}(S^1, \mathfrak{L})$$

is sufficiently $C^0(S^1, B^{s',p}(\Sigma))$ close to

$$\gamma = \widehat{r_\Sigma}(B(t), \Phi(t)) \in \mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L})$$

so that we can find a unique $z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L})$ near 0 satisfying

$$\gamma = \mathcal{E}_{\gamma_0}(z). \tag{12.8}$$

Rewriting this in terms of the map $\mathcal{E}^1_{\gamma_0}$ in Theorem 10.9, we thus have

$$\gamma = \gamma_0 + z + \mathcal{E}^1_{\gamma_0}(z), \qquad z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L}). \tag{12.9}$$

In other words, we have placed γ in the range of the chart map $\mathcal{E}_{\gamma_0}$ centered at the smooth configuration γ_0. Altogether, the interior equations (11.13), the Neumann boundary condition on b (11.14), and the boundary condition (12.9) yield the following form for the full system of Seiberg-Witten equations with Lagrangian boundary conditions:

$$\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_0\right)(b, \phi, \xi) = N_{(A_0,\Phi_0)}(b, \phi, \xi) - SW_4(A_0, \Phi_0) \tag{12.10}$$

$$(b, \phi)|_{S^1\times\Sigma} = z + \mathcal{E}^1_{\gamma_0}(z), \qquad z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p}(S^1, \mathfrak{L}) \tag{12.11}$$

$$*b|_{S^1\times\Sigma} = 0. \tag{12.12}$$

Recall that $N_{(A_0,\Phi_0)}$ is a quadratic multiplication map and $SW_4(A_0, \Phi_0)$ is a smooth term since (A_0, Φ_0) is smooth. Since we are only interested in regularity near the boundary, it suffices to gain regularity for a smooth truncation of (b, ϕ, ξ) with support near the boundary. Thus, define

$$(b_0, \phi_0, \xi_0) = \chi(b, \phi, \xi) \tag{12.13}$$

where χ is a smooth cutoff function supported in a collar neighborhood $S^1 \times [0, 1] \times \Sigma$ of the boundary, with $\chi \equiv 1$ on $S^1 \times [0, 1/2] \times \Sigma$ and $\chi \equiv 0$ on outside of $S^1 \times [0, 3/4] \times \Sigma$. Thus, via the notation of Section 3, we have

$$(b_0, \phi_0, \xi_0) \in H^{1,p}\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}).$$

Step Two: We will gain regularity for (b_0, ϕ_0, ξ_0) in the Σ directions from the equations (12.10)-(12.12). For this, we will use the linear theory on L^2 spaces developed in the previous section. The main idea is simple. The boundary condition (12.11) and (12.12) is essentially a perturbation of the linear boundary condition studied in Section 3. Indeed, Theorems 11.2 and 11.6 give us elliptic estimates when the nonlinear term $\mathcal{E}^1_{\gamma_0}(z)$ in (12.11) is absent, since then the boundary condition (12.11) satisfies the linear boundary conditions of Thereoms 11.2 and 11.6, where $L(t) = T_{\gamma_0(t)}\mathrm{Maps}(S^1, \mathfrak{L})$. Moreover, since (b_0, ϕ_0, ξ_0) is supported on $S^1 \times [0, 3/4] \times \Sigma$, the (b_0, ϕ_0, ξ_0) satisfy any boundary condition on $\{1\} \times \Sigma$, so that we may use Theorem 11.6 for any suitable "dummy" boundary condition L_1. With the nonlinear term in (12.11) however, we use the fact that $\mathcal{E}^1_{\gamma_0}$ is smoothing in the Σ directions, as given by Theorem 10.9. Thus, we are able to gain regularity in the Σ directions using the inhomogeneous elliptic estimate Corollary 11.9.

In detail, we first have to embed the $p \neq 2$ Besov spaces into the $p = 2$ Besov spaces of Section 3. For this, we use the embedding $B^{s,p}(X) \hookrightarrow B^{s-\epsilon,2}(X)$ for any $s \in \mathbb{R}$, $p > 2$, and $\epsilon > 0$, on any compact manifold X (see Part IV). In particular, we have $B^{1-1/p,p}(S^1 \times \Sigma) \hookrightarrow B^{1-1/p-\epsilon,2}(S^1 \times \Sigma)$. Consequently, using Theorem 10.9, we have

$$\mathcal{E}^1_{\gamma_0}(z) \in \mathrm{Maps}^{(1-1/p,1-1/p-\epsilon),p}(S^1, \mathcal{T}_\Sigma) \hookrightarrow \mathrm{Maps}^{(1-1/p-\epsilon,1-1/p-\epsilon),2}(S^1, \mathcal{T}_\Sigma). \tag{12.14}$$

With $L(t) = T_{\gamma_0(t)}\mathrm{Maps}(S^1, \mathfrak{L})$ and $L_1 \subset \mathcal{T}_{\Sigma_1}$ as in Corollary 11.9, we have a projection

$$\Pi_{\gamma_0} : \mathrm{Maps}^{(k+1/2,s),2}(S^1, \widetilde{\mathcal{T}}_{\Sigma_0} \oplus \widetilde{\mathcal{T}}_{\Sigma_1}) \circlearrowleft \tag{12.15}$$

defined by (11.79) in Corollary 11.9. By definition of z in (12.11) and construction of Π, we have that $(z, 0, \xi|_{S^1\times\Sigma}) \in \mathrm{Maps}^{1-1/p,p}(S^1, \widetilde{\mathcal{T}}_\Sigma)$ satisfies

$$\Pi_{\gamma_0}(z, 0, \xi|_{S^1\times\Sigma}) = 0. \tag{12.16}$$

Let $x = (b, \phi, \xi)$ and $x_0 = (b_0, \phi_0, \xi_0)$. Thus applying the estimate (11.80) to $x_0 = (b_0, \phi_0, \xi_0)$ and using equations (12.10)-(12.12) and (12.16), we have

$$\begin{aligned}\|x_0\|_{B^{(k+1,s),2}((S^1\times[0,1])\times\Sigma)} \leq C\bigg(& \|N_{(A_0,\Phi_0)}(x)\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)} + \|\Pi\mathcal{E}^1_{\gamma_0}(z)\|_{B^{(k+1/2,s),2}(S^1\times(\Sigma\cup\Sigma_1))} \\ & + \|x\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)} + \|SW_4(A_0,\Phi_0)\|_{B^{(k,s),2}((S^1\times[0,1])\times\Sigma)}\bigg).\end{aligned} \tag{12.17}$$

for all $k \geq 1$ and $s \in [0,1]$ such that the right-hand side is finite (see also Remark 11.3).

First, let $k = 0$ and $s = 1$ in (12.17). Since $x \in H^{1,p}((S^1 \times [0,1]) \times \Sigma) \hookrightarrow B^{(0,1),2}((S^1 \times [0,1]) \times \Sigma)$, this means we always have control of the lower order third term of (12.17). Furthermore, since $H^{1,p}(S^1 \times Y)$ is an algebra, we have

$$N(x) \in H^{1,p}(S^1 \times [0,1] \times \Sigma) \hookrightarrow B^{(0,1),2}(S^1 \times [0,1] \times \Sigma). \tag{12.18}$$

since $p > 4$. The final term to control is the boundary term, for which we have

$$\Pi(\mathcal{E}^1_{\gamma_0}(z)) \in \mathrm{Maps}^{(1/2,1),2}(S^1, \widetilde{\mathcal{T}}). \tag{12.19}$$

Here, we used (12.14) and the embedding

$$\mathrm{Maps}^{(1-1/p-\epsilon,1-1/p-\epsilon),2}(S^1, \mathcal{T}_\Sigma) \hookrightarrow \mathrm{Maps}^{(1/2,1),2}(S^1, \widetilde{\mathcal{T}}_\Sigma),$$

which follows since $p > 4$ and $\epsilon > 0$ can be chosen small.

Thus, we see that $\|\Pi\mathcal{E}^1_{\gamma_0}(z)\|_{B^{(1/2,1),2}(S^1\times\Sigma)}$ is bounded thanks to the smoothing property of $\mathcal{E}^1_{\gamma_0}(z)$. Furthermore, the boundedness of Π and the preceding analysis imply that

$$\begin{aligned}\|\Pi\mathcal{E}^1_{\gamma_0}(z)\|_{B^{(1/2,1),2}(S^1\times\Sigma)} &\leq \|\mathcal{E}^1_{\gamma_0}(z)\|_{B^{(1-1/p,1-1/p-\epsilon),p}(S^1\times\Sigma)} \\ &\leq \mu_{\gamma_0}(\|z\|_{B^{1-1/p,p}(S^1\times\Sigma)}) && (12.20) \\ &\leq \mu_{\gamma_0}(\|x\|_{H^{1,p}(S^1\times\Sigma)}) && (12.21)\end{aligned}$$

where $\mu = \mu_{\gamma_0} : \mathbb{R}^+ \to \mathbb{R}^+$ is some continuous nonlinear function depending on γ_0 with $\mu(0) = 0$. It suffices to prove (12.20), since the second line is just the trace theorem. The main point is that even though Theorem 10.9 tells us that $\mathcal{E}^1_{\gamma_0}(z) \in \mathrm{Maps}^{(1-1/p,1-1/p-\epsilon),p}(I, \mathcal{T}_\Sigma)$ given that $z \in \mathrm{Maps}^{1-1/p,p}(I, \mathcal{T}_\Sigma)$, we want an estimate in term of norms, as expressed by (12.20). However, we shall leave it to the reader to check that one can indeed obtain a norm estimate if one follows through all the various operators and constructions used in defining

$\mathcal{E}^1_{\gamma_0}$. In a few words, we obtain norm estimates because all estimates we perform along the way are derived from multiplication theorems, elliptic bootstrapping, interpolation, etc., all of which provide explicit norm dependent estimates. Hence, this proves (12.20) and therefore (12.21).

Notation: In what follows, we write $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ to denote any continuous nonlinear function. Any subscripts on μ will be quantities which μ depends on which we wish to make explicit. The precise form of μ is immaterial and may change from line to line.

Altogether, we have from (12.17), (12.18), and (12.21) that

$$\|(b_0, \phi_0, \xi_0)\|_{B^{(1,1),2}((S^1\times[0,1])\times\Sigma)} \leq \mu(\|(b,\phi,\xi)\|_{H^{1,p}(S^1\times Y)}) + \|SW_4(A_0,\Phi_0)\|_{B^{(0,1),2}((S^1\times[0,1])\times\Sigma)}. \tag{12.22}$$

Thus, we have gained a whole derivative in the Σ direction, albeit only in L^2 and not with integrability p.

Step Three: To gain regularity in the temporal and normal directions S^1 and $[0,1]$, respectively, we use the methods of [52] which studies Cauchy-Riemann equations with values in a Banach space. Here, the main results we need are summarized in Theorem 16.2, which is a refinement of [52, Theorem 1.2] to our situation. Let us set up the notation for this analysis.

We have by definition

$$(b,\phi,\xi) \in H^{1,p}\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) = H^{1,p}\mathrm{Maps}\left(S^1, \Omega^1([0,1]\times\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{[0,1]\times\Sigma}) \oplus \Omega^0([0,1]\times\Sigma; i\mathbb{R})\right).$$

Let

$$K = S^1 \times [0,1] \tag{12.23}$$

and let (t,v) be the corresponding temporal and normal coordinates on K. Observe that we have the identification

$$\mathrm{Maps}(S^1, \widetilde{\mathcal{T}}_{[0,1]\times\Sigma}) \cong \Gamma\left(K; \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma) \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R})\right) \tag{12.24}$$

$$= \Gamma(K, \widetilde{\mathcal{T}}_\Sigma), \tag{12.25}$$

via the restriction map

$$r_v : \widetilde{\mathcal{T}}_{[0,1]\times\Sigma} \to \widetilde{\mathcal{T}}|_{\{v\}\times\Sigma} \cong \widetilde{\mathcal{T}}_\Sigma = \Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma) \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R}) \tag{12.26}$$

induced by the restriction map r in (11.19) for each $v \in [0,1]$.

Thus, we can regard (b,ϕ,ξ), a path from S^1 into $\widetilde{\mathcal{T}}_{[0,1]\times\Sigma}$, as a map from K to $\widetilde{\mathcal{T}}_\Sigma$, in the appropriate function space topologies. Since we have the embeddings

$$H^{1,p}(K\times\Sigma) \hookrightarrow C^0(S^1, B^{1-1/p,p}([0,1]\times\Sigma)) \hookrightarrow C^0(K, B^{1-2/p,p}(\Sigma)),$$

and

$$B^{1-2/p,p}(\Sigma) \hookrightarrow L^p(\Sigma)$$

for $p > 2$, we can thus regard

$$(b, \phi, \xi) \in H^{1,p}(K; L^p\widetilde{\mathcal{T}}_\Sigma). \tag{12.27}$$

The space $H^{1,p}(K; L^p(\widetilde{\mathcal{T}}_\Sigma)$ is the Sobolev space of $H^{1,p}(K)$ functions with values in the Banach space $L^p\widetilde{\mathcal{T}}_\Sigma$. As it turns out, we want to consider the larger space $H^{1,2}(K; L^2\widetilde{\mathcal{T}}_\Sigma) = B^{1,2}(K; L^2\widetilde{\mathcal{T}}_\Sigma)$, and we will instead regard

$$(b, \phi, \xi) \in B^{1,2}(K; L^2\widetilde{\mathcal{T}}_\Sigma). \tag{12.28}$$

We want to use (12.28) instead of (12.27) because the regularity in Σ we gained in Step Two were with $p = 2$ Besov spaces. This gain in regularity becomes essential when we reformulate the Seiberg-Witten equations as a nonlinear Cauchy-Riemann equation as we now explain.

Near the boundary, the Cauchy-Riemann operator occuring for us arises from the t and v derivatives of the operator $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ occurring in (12.10). Indeed, $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ is a Dirac operator, and so near the boundary where the metric is of the product form $g^2 = dt^2 + dv^2 + g^2_{\Sigma,v}$, we can write

$$\frac{d}{dt} + \widetilde{\mathcal{H}}_0 = \frac{d}{dt} + J\frac{d}{dv} + D_\Sigma, \tag{12.29}$$

where $J : \widetilde{\mathcal{T}}_\Sigma \to \widetilde{\mathcal{T}}_\Sigma$ is a smooth, bundle automorphism satisfying $J^2 = -1$, and D_Σ is a v-dependent differential operator acting on $\widetilde{\mathcal{T}}_\Sigma$. Since we have gained regularity in the Σ directions for (b, ϕ, ξ) in the previous step, the Σ derivatives of $\frac{d}{dt} + \widetilde{\mathcal{H}}_0$ can be absorbed into (b, ϕ, ξ) and moved to the right-hand-side of (12.10). Thus, (12.10) yields the semilinear Cauchy-Riemann equation

$$\left(\frac{d}{dt} + J\frac{d}{dv}\right)(b, \phi, \xi) = -D_\Sigma(b, \phi, \xi) + N_{(A_0,\Phi_0)}(b, \phi, \xi) - SW_4(A_0, \Phi_0). \tag{12.30}$$

In this setting, we reinterpret the boundary conditions (12.11) and (12.12) as follows. Recall that the configuration $(A, \Phi) = (b, \phi, \xi) + (A_0, \Phi_0)$ and the smooth configuration (A_0, Φ_0) both satisfy the Lagrangian boundary conditions (12.2). Thus, both $r_\Sigma(b(t), \phi(t)) + r_\Sigma(B_0(t), \Phi_0(t))$ and $r_\Sigma(B_0(t), \Phi_0(t))$ are elements of $L^2\mathfrak{L} \supset \mathcal{L}^{1-2/p,p}$ for every $t \in S^1$, by the above analysis. Observe that for any pair of configurations $u, u_0 \in \mathfrak{C}^{0,2}(\Sigma)$, we have $u - u_0 \in \mathcal{T}^{0,2}_\Sigma$ since $\mathfrak{C}^{0,2}(\Sigma)$ is an affine space modeled on $\mathcal{T}^{0,2}_\Sigma$. In particular, if $u_0 = (0, 0)$ is the zero connection and zero spinor[19], we can regard $\mathfrak{C}^{0,2}(\Sigma) = \mathcal{T}^{0,2}_\Sigma$. In particular, we may regard $L^2\mathfrak{L} \subset \mathcal{T}^{0,2}_\Sigma$ and we may regard

$$L^2\widetilde{\mathfrak{L}} := L^2\mathfrak{L} \times 0 \times L^2\Omega^0(\Sigma; i\mathbb{R}) \tag{12.31}$$

as a subset of $\widetilde{\mathcal{T}}^{0,2}_\Sigma$. Moreover, we may regard $r_\Sigma(B_0(t), \Phi_0(t))$ as a continuous path in $\widetilde{\mathcal{T}}^{0,2}_\Sigma$.

With these identification, the boundary conditions (12.11) and (12.12) can be expressed along the boundary of K, i.e., at $v = 0$, as

$$r_0(b(t), \phi(t), \xi(t)) + r_\Sigma(B_0(t), \Phi_0(t)) \in L^2\widetilde{\mathfrak{L}}, \qquad \text{for all } t \in S^1. \tag{12.32}$$

[19]For convenience, we assume the spinor bundle $\mathcal{S}_\Sigma$ on Σ is trivial. This merely simplifies the notation since the reference configuration u_0 can be chosen to be zero.

Thus, the boundary condition (12.32) captures the tangential Lagrangian boundary condition via the $L^2\mathfrak{L}$ factor of $L^2\widetilde{\mathfrak{L}}$, and it captures the Neumann boundary condition on b via the remaining $0 \times L^2(\Omega^0(\Sigma; i\mathbb{R}))$ factor of $L^2\widetilde{\mathfrak{L}}$.

Altogether, we have a semilinear Cauchy-Riemann equation (12.30) with values in a Banach space $\widetilde{\mathcal{T}}^{0,2}_\Sigma$ and with boundary condition specified by (12.32). We can apply Theorem 16.2 when the boundary condition (12.32) is given by a Banach manifold modeled on a closed subspace of an L^p space. In [34], we studied the L^p closure of $\mathfrak{L}$ for $p \geq 2$ and showed that, while we do not know if globally $L^p\mathfrak{L}$ is a manifold, we know that locally the chart maps E_{u_0} for $\mathfrak{L}$ at a smooth configuration $u_0 \in \mathfrak{L}$ are bounded in the L^p topology (see Corollary 4.16). More precisely, for every $2 \leq p < \infty$, there exists an $L^p(\Sigma)$ neighborhood U of 0 in $L^pT_{u_0}\mathfrak{L}$ containing an L^p open ball, such that E_{u_0} extends to a bounded map

$$E_{u_0} : U \to L^p\mathfrak{C}(\Sigma) \tag{12.33}$$

which is a diffeomorphism onto its image. Moreover, because of the trace map

$$H^{1,p}(S^1 \times Y) \hookrightarrow C^0(K, B^{1-2/p,p}(\Sigma))$$

and because $\mathfrak{L}^{1-2/p,p} = B^{1-2/p,p}\mathfrak{L}$ is globally a smooth embedded submanifold of $\mathfrak{C}^{1-2/p,p}(\Sigma)$ (by [34] since $p > 4$), we know that the path

$$\Big(t \mapsto r_\Sigma(b(t), \phi(t)) + r_\Sigma(B_0(t), \Phi_0(t))\Big) \in C^0(S^1, \mathfrak{L}^{1-2/p,p}) \tag{12.34}$$

forms a continuous path in $\mathfrak{L}^{1-2/p,p}$, and hence on a small time interval $I \subset S^1$, the path lies in a single coordinate chart of a fixed configuration $u_0 \in \mathfrak{L}^{1-2/p,p}$ which we may take to be smooth. In fact, we may as well take $u_0 = r_\Sigma(B_0(t_0), \Psi_0(t_0))$ for some fixed $t_0 \in I$. Thus, we may replace (12.32), which may not be a manifold boundary condition in general, with

$$r_0(b(t), \phi(t), \xi(t)) + r_\Sigma(B_0(t), \Phi_0(t)) \in \widetilde{E_{u_0}(U)}, \qquad U \subset L^2T_{u_0}\mathfrak{L}, \text{ for all } t \in I \tag{12.35}$$

where U is an $L^2(\Sigma)$ open neighborhood of $0 \in L^2T_{u_0}\mathfrak{L}$ and

$$\widetilde{E_{u_0}(U)} := E_{u_0}(U) \times 0 \times L^2\Omega(\Sigma; i\mathbb{R}).$$

By the above remarks, (12.35) is a manifold boundary condition, since $E_{u_0}(U)$ is a submanifold of $L^2\mathfrak{C}(\Sigma)$. In effect, we have simply replaced a neighborhood of $u_0 \in \mathfrak{L}^{1-2/p,p} \subset \mathfrak{C}^{1-2/p,p}(\Sigma)$ with the larger L^2 neighborhood $E_{u_0}(U) \subset L^2\mathfrak{C}(\Sigma)$. Moreover, since $\mathfrak{L} \subseteq \mathfrak{C}(\Sigma)$ is a Lagrangian submanifold, then $E_{u_0}(U)$ is a Lagrangian submanifold of $L^2\mathfrak{C}(\Sigma)$. Thus, $\widetilde{E_{u_0}(U)}$ is a product Lagrangian submanifold of

$$L^2(\mathfrak{C}(\Sigma) \times \Omega^0(\Sigma; i\mathbb{R}) \times \Omega^0(\Sigma; i\mathbb{R})), \tag{12.36}$$

where the symplectic form on (12.36) is given by the product symplectic form (11.28). The time interval I in (12.35) is chosen small enough so that the configuration $r_0(b(t), \phi(t), \xi(t)) + r_\Sigma(B_0(t), \Phi_0(t))$ remains inside the product chart $\widetilde{E_{u_0}(U)}$. To simplify the below analysis, we can just suppose $I = S^1$. Otherwise, we can cover S^1 with small time intervals and sum up the estimates on each interval all the same.

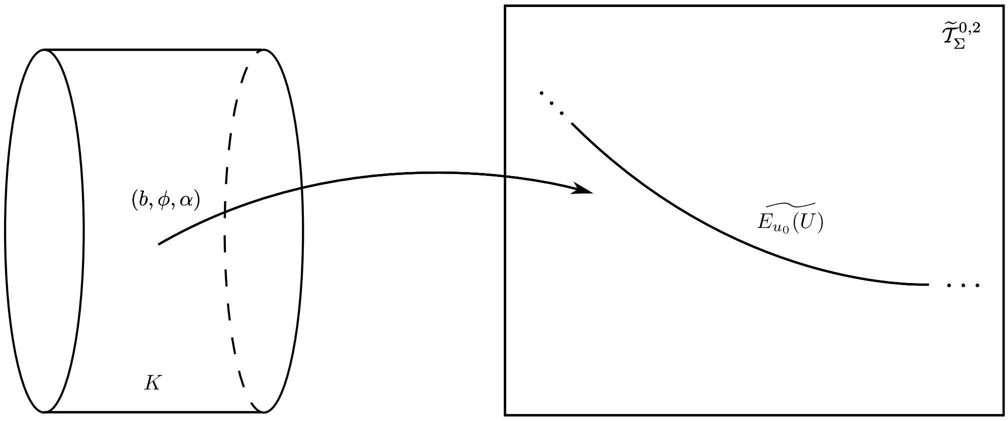

Figure III-2: The configuration (b, ϕ, ξ) is a function on $K = S^1 \times [0, 1]$ with values in the Banach space $\widetilde{\mathcal{T}}^{0,2}_\Sigma$.

Altogether, equations (12.30) and (12.35) form a Cauchy-Riemann equation for a configuration with values in a Banach space supplemented with Lagrangian boundary conditions. Here, the Lagrangian submanifold $\widetilde{E_{u_0}(U)}$ is modeled on a closed subspace of an L^2 space. Furthermore, it is an analytic Banach submanifold of $\widetilde{\mathcal{T}}^{0,2}_\Sigma$. This is because the chart map E_{u_0}, by Theorem 4.8, is constructed from the local straightening map $F^{-1}_{(B_0,\Psi_0)}$, where $(B_0, \Psi_0) \in \mathcal{M}$ satisfies $r_\Sigma(B_0, \Psi_0) = u_0$. As discussed in the proof of Lemma 10.8, the map $F^{-1}_{(B_0,\Psi_0)}$ is analytic, which implies the analyticity of E_{u_0}.

With all our current function spaces being Sobolev spaces, we now apply Theorem 16.2, where so as to not confuse the value of p in our present situation with that in Theorem 16.2, we let p' denote what is p in Theorem 16.2. So letting $X = L^2\widetilde{\mathcal{T}}_\Sigma$, $k = 1$, $p' = 2$, and $q' = q = p > 4$, the hypotheses of Theorem 16.2(i) are satisfied, and we obtain

$$(b, \phi, \xi) \in B^{2,2}(K, L^2\widetilde{\mathcal{T}}_\Sigma), \tag{12.37}$$

i.e., we have gained a whole derivative in the K directions. We can take $q' = q = p$, since

$$(b, \phi, \xi) \in H^{1,p}(K, L^p\widetilde{\mathcal{T}}_\Sigma), \tag{12.38}$$

and $L^p\widetilde{\mathcal{T}}_\Sigma \subset X$, since $p > 4$. Furthermore, the elliptic estimate (16.6) implies

$$\begin{aligned} \|(b_0, \phi_0, \xi_0)\|_{B^{2,2}(K;L^2\widetilde{\mathcal{T}}_\Sigma)} &\leq \mu_{(A_0,\Phi_0)}\Big(\|D_\Sigma(b_0, \phi_0, \xi_0)\|_{B^{1,2}(K;L^2\widetilde{\mathcal{T}}_\Sigma)} + \|N_{(A_0,\Phi_0)}(b, \phi, \xi)\|_{B^{1,2}(K;L^2\widetilde{\mathcal{T}}_\Sigma)} \\ &\qquad + \|(b, \phi, \xi)\|_{B^{1,2}(K;L^2\widetilde{\mathcal{T}}_\Sigma)} + \|SW_4(A_0, \Phi_0)\|_{B^{1,2}(K;L^2\widetilde{\mathcal{T}}_\Sigma)}\Big) \\ &\leq \mu_{(A_0,\Phi_0)}\Big(\|(b, \phi, \xi)\|_{H^{1,p}(S^1\times Y)} + \|SW_4(A_0, \Phi_0)\|_{B^{1,2}(K;L^2\widetilde{\mathcal{T}}_\Sigma)}\Big) \end{aligned} \tag{12.39}$$

for some nonlinear function $\mu_{(A_0,\Phi_0)}$. Here (b_0, ϕ_0, ξ_0) plays the role of $u - u_0$ in (16.6).

Summarizing, by using Theorem 16.2 we deduced (12.39) and gained regularity in the K directions, i.e., we now have two derivatives in the $S^1 \times [0,1]$ directions in L^2. Combined with the estimates from Step Two, where we gained regularity in just the Σ directions, we see that we have gained a whole derivative in *all* directions, i.e., $(b_0, \phi_0, \xi_0) \in B^{2,2}\widetilde{\mathcal{T}}_{[0,1]\times\Sigma}$. Combined with interior regularity, altogether we have the elliptic estimate

$$\|(b,\phi,\xi)\|_{B^{2,2}(S^1\times Y)} \leq \mu_{(A_0,\Phi_0)}\Big(\|(b,\phi,\xi)\|_{H^{1,p}(S^1\times Y)}\| + \|SW_4(A_0,\Phi_0)\|_{H^{1,p}(S^1\times Y)}\Big). \quad (12.40)$$

on all of $S^1 \times Y$.

Step Four: From the previous steps, our configuration $(A,\Phi) \in H^{1,p}(S^1 \times Y)$, which we redefined by a gauge transformation so that it is in Coulomb-Neumann gauge with respect to (A_0, Φ_0), is in $B^{2,2}(S^1 \times Y)$. Moreover, we have the elliptic estimate (12.40). Proceeding as in the previous two steps, we want to bootstrap and show that $(b,\phi,\xi) = (A,\Phi) - (A_0,\Phi_0)$ is in $B^{k,2}(S^1 \times Y)$ for all $k \geq 2$, which will prove the theorem. Unfortunately, in our first step when we want to bootstrap from $B^{2,2}(S^1 \times Y)$ to $B^{3,2}(S^1 \times Y)$, the space $B^{2,2}(S^1 \times Y)$ is not strictly stronger than the original space $H^{1,p}(S^1 \times Y)$, i.e., we do not have an embedding $B^{2,2}(S^1 \times Y) \hookrightarrow H^{1,p}(S^1 \times Y)$, since $p > 4$. Thus, we will need to work with the mixed topology $H^{1,p}(S^1 \times Y) \cap B^{2,2}(S^1 \times Y)$. This is the cause for the rather bizarre looking Thereom 10.10.

We first start off by increasing the regularity of $E^1_{\gamma_0}(z)$ in (12.9). Indeed, since we now have $(A,\Phi) \in H^{1,p}(S^1 \times Y) \cap B^{2,2}(S^1 \times Y)$, then $\gamma \in \mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L}) \cap \mathrm{Maps}^{3/2,2}(S^1, \mathfrak{L})$. Thus, we can write

$$\gamma = \gamma_0 + z + \mathcal{E}^1_{\gamma_0}(z), \qquad z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p,p}(S^1, \mathfrak{L}) \cap \mathrm{Maps}^{3/2,2}(S^1, \mathfrak{L}), \quad (12.41)$$

and by Theorem 10.10 with $s_2 = 0$, we have

$$\mathcal{E}^1_{\gamma_0}(z) \in \mathrm{Maps}^{(1-1/p,1-1/p-\epsilon),p}(S^1, \mathcal{T}_\Sigma) \cap \mathrm{Maps}^{(3/2,1/2),2}(S^1, \mathcal{T}_\Sigma), \quad (12.42)$$

Here, we have assumed that γ_0 is sufficiently close to γ in $C^0(I, B^{s',p})$, using the *same* s' we used in Step Two when we applied Theorem 10.9 to (12.9), so that we can place γ in the chart map (12.41) using Theorem 10.10, which is stronger than (12.9). (We could of course redefine γ_0 at this step by moving it closer to γ if necessary.) Here, to perform the above step, it is crucial that in both these theorems, the size of the chart maps (the radius δ which appears) depends locally uniformly with respect to the $B^{s',p}$ topology, which is very weak since we can choose any $1/2 < s' < s - 2/p$ (since $s_2 = 0$).

Now if we focus on the second factor of (12.42), we see that $\mathcal{E}^1_{\gamma_1}(z)$ smooths by $1/2$ a derivative in the Σ directions. Plugging this into (12.17) and proceeding as before, we find

$$\begin{aligned}\|(b,\phi,\xi)\|_{B^{(2,1/2),2}((S^1\times[0,1])\times\Sigma)} &\le \mu_{(A_0,\Phi_0)}\Big(\|(b,\phi,\xi)\|_{B^{2,2}(S^1\times Y)} + \|(b,\phi,\xi)\|_{H^{1,p}(S^1\times Y)} \\ &\qquad \|SW_4(A_0,\Phi_0)\|_{B^{2,2}(S^1\times Y)}\Big), \\ &\le \mu_{(A_0,\Phi_0)}\Big(\|(b,\phi,\xi)\|_{H^{1,p}(S^1\times Y)} + \\ &\qquad \|SW_4(A_1,\Phi_1)\|_{B^{2,2}(S^1\times Y)}\Big), \end{aligned} \tag{12.43}$$

where we use that $\|(b,\phi,\xi)\|_{B^{2,2}(S^1\times Y)}$ is controlled by $\|(b,\phi,\xi)\|_{H^{1,p}(S^1\times Y)}$ via (12.40).

Estimate (12.43) is insufficient however since we want to gain a full derivative in the Σ direction, i.e., we want control of $B^{(2,1),2}((S^1\times[0,1])\times\Sigma)$ instead of $B^{(2,1/2),2}((S^1\times[0,1])\times\Sigma)$. Thus, we repeat the above steps again, where we replace $\mathrm{Maps}^{3/2,2}(S^1,\mathfrak{L})$ with $\mathrm{Maps}^{(3/2,s_2),2}(S^1,\mathfrak{L})$, with $s_2=1/2$. Using the same reasoning as before (and assuming (A_0,Φ_0) is sufficiently $C^0(S^1,B^{s',p}(\Sigma))$ close to (A,Φ) on the boundary), by Theorem 10.10, we obtain

$$\mathcal{E}^1_{\gamma_0}(z)\in \mathrm{Maps}^{(1-1/p,1-1/p-\epsilon),p}(S^1,\mathcal{T}_\Sigma)\cap \mathrm{Maps}^{(3/2,1),2}(S^1,\mathcal{T}_\Sigma), \tag{12.44}$$

thereby improving the gain in Σ regularity from $1/2$ to 1. Doing this, we now gain a whole derivative in the Σ directions:

$$\begin{aligned}\|(b,\phi,\xi)\|_{B^{(2,1),2}((S^1\times[0,1])\times\Sigma)} &\le \mu_{(A_0,\Phi_0)}\Big(\|(b,\phi,\xi)\|_{H^{1,p}(S^1\times Y)} + \\ &\qquad \|SW_4(A_0,\Phi_0)\|_{B^{2,2}(S^1\times Y)}\Big). \end{aligned} \tag{12.45}$$

Having gained a whole derivative in the Σ directions, we can proceed to Step Three and gain regularity in the $S^1\times[0,1]$ directions. Here, we need to choose our parameters in Theorem 16.2 appropriately. In the same way that we needed to proceed in two steps to gain a whole derivative in the Σ directions, we will also need to proceed in two steps to gain a whole derivative in the $S^1\times[0,1]$ directions as well. First, we let $k=1$. By the above, we have $x=(b,\phi,\xi)$ belongs to the space $H^{(2,1),2}(K\times\Sigma)$ in addition to belonging to the space $H^{1,p}(K\times\Sigma)$. Thus, $D_\Sigma\in H^{2,2}(K\times\Sigma)\subset H^{1,4}(K\times\Sigma)$. Using the multiplication theorem Theorem 13.18, we have that $H^{(2,1),2}(K\times\Sigma)\cap L^\infty$ is an algebra. In particular, $x\#x\in H^{(2,1),2}(K\times\Sigma)\subset H^{1,4}(K\times\Sigma)$, where $\#$ denote any pointwise multiplication map. It follows that x satisfies a Cauchy-Riemann equation (16.3), where G has the same regularity as $D_\Sigma x + x\#x\in H^{1,4}(K\times\Sigma)$. Thus, thinking of G as taking values in

$$X := L^{p_0}\widetilde{\mathcal{T}}_\Sigma$$

for some $p_0=2+\epsilon$, where $\epsilon>0$ is small, we have

$$G\in H^{1,4}(K,X). \tag{12.46}$$

Thus, we may apply Theorem 16.2 with $p'=p_0$, $q=4$, and $q'=p>4$ (recall that p' is the value of the dummy variable "p" in Theorem 16.2, to distinguish it from our present value of p). Note that when we change from the Banach space $L^2\widetilde{\mathcal{T}}_\Sigma$ to $X=L^{p_0}\widetilde{\mathcal{T}}_\Sigma$, we must also consider for the Lagrangian boundary values of x the locally embedded L^{p_0} charts associated with $\mathfrak{L}$, instead of L^2 charts as before. This is possible since $x \in C^0(K, L^p\widetilde{\mathcal{T}}_\Sigma)$, and $L^p\widetilde{\mathcal{T}}_\Sigma \subset L^{p_0}\widetilde{\mathcal{T}}_\Sigma$ since $p > p_0$. In any event, we apply Theorem 16.2 and obtain $x \in H^{2,p_0}(K, X)$. On the other hand, since

$$x \in H^{(2,1),2}(K \times \Sigma) \subset H^{2,4}(\Sigma, L^4(K)) \subset H^{2,p_0}(\Sigma, L^{p_0}(K)),$$

it follows that $x \in H^{2,p_0}(K \times \Sigma)$.

This implies we have improved the integrability of (A, Φ) from $H^{1,p}(S^1 \times Y) \cap H^{2,2}(S^1 \times Y)$ to $H^{1,p}(S^1 \times Y) \cap H^{2,p_0}(S^1 \times Y)$, with $p_0 > 2$. This extra integrability now allows us to increase the regularity of x by applying Theorem 16.2 with $k = 2$. Here, we let $p' = q = 2$, and $q' = p_0$. Observe that $q' = 2$ does not work, which is why we needed the above step. Doing this gives us a configuration in $H^{3,2}(S^1 \times Y) = B^{3,2}(S^1 \times Y)$ which is strictly stronger than $H^{1,p}(S^1 \times Y)$ for p close to 4.

We can now continue bootstrapping as above, using Theorem 10.10 and estimate (11.80) as above to gain Σ regularity, and then Theorem 16.2 to gain $S^1 \times [0,1]$ regularity. Each time, we apply Theorem 10.10 to gain a full derivative in the Σ directions, and then we apply Theorem 16.2 once to gain a whole derivative in the $S^1 \times [0,1]$ directions. Indeed, our function spaces are now sufficiently regular that we can apply Theorem 10.10 to gain a whole derivative in the Σ directions ($s > 3/2$ so $\epsilon' = 0$ in the theorem), and we can apply Theorem 16.2 to gain one whole derivative in the $S^1 \times [0,1]$ directions in one step without having to first bootstrap the integrability of our configuration as in the above. Together, these steps gain for us a whole derivative in all directions.

Altogether, we have shown the following. Pick any smooth reference connection A_{ref} and redefine (A, Φ) by a gauge transformation that places A in Coulomb-Neumann gauge with respect to A_{ref}. Then finding smooth (A_0, Φ_0), satisfying the Lagrangian boundary conditions, that is sufficiently $H^{s,p}(S^1 \times Y)$ close to (A, Φ), with $s > 1/2 + 2/p$, then for every $k \geq 2$, we have the estimate

$$\|(A - A_0, \Phi - \Phi_0)\|_{B^{k,2}(S^1 \times Y)} \leq \mu_{k,(A_0,\Phi_0)}\Big(\|(A - A_0, \Phi - \Phi_0)\|_{H^{1,p}(S^1 \times Y)} + \|SW(A_0, \Phi_0)\|_{B^{k-1,2}(S^1 \times Y)}\Big), \qquad (12.47)$$

where $\mu_k = \mu_{k,(A_0,\Phi_0)}$ is a continuous nonlinear function depending on k and (A_0, Φ_0). This estimate proves the theorem. □

There is no obstacle to extending the above result to the equations on $\mathbb{R} \times Y$:

Proof of Theorem A: We can cover $\mathbb{R} \times Y$ with a sequence of compact manifolds with boundary X_k, $k \in \mathbb{Z}$, with $X_i \cap X_j = \emptyset$ unless $|i - j| = 1$. Here, each X_k is assumed to contain an open cylinder $I_k \times Y$ for some open interval $I_k \subset \mathbb{R}$. We can pick a nearby smooth configuration (A_0, Φ_0), as in the proof of Theorem 12.1, on all of $\mathfrak{C}(\mathbb{R} \times Y)$. We then place A in Coulomb-Neumann gauge with respect to A_0. We can do this on all of $\mathbb{R} \times Y$ since we can do so on each compact manifold X_k, and then we can patch the gauge transformations together to get a single-gauge transformation $g \in H^{2,p}_{\mathrm{loc}}(\mathbb{R} \times Y)$. There is no issue with patching, since the gauge group is abelian, and all the local gauge transformations on X_k are elements of the identity component of the gauge group. We let (b, ϕ, ξ) be $(A, \Phi) - (A_0, \Phi_0)$

as before. On each X_k near the boundary, we can find a compactly supported cutoff function $\chi_k : I_k \times [0,1) \times \Sigma \to \mathbb{R}$, with $\chi_k \equiv 1$ on $I'_k \times [0,1/2] \times \Sigma$, where $I'_k \subseteq I_k$ has compact support. We can then repeat the previous steps for the compactly supported configuration $\chi_k(b,\phi,\xi)$, since it satisfies the system

$$\left(\frac{d}{dt} + \widetilde{\mathcal{H}}_0\right)\chi_k(b,\phi,\xi) = N_k(b,\phi,\xi) - \chi_k SW_4(A_0,\Phi_0) \tag{12.48}$$

$$\chi_k(b,\phi)|_{S^1\times\Sigma} = \chi_k z + \chi_k \mathcal{E}^1_{\gamma_0}(z), \qquad z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p}(S^1,\mathfrak{L}) \tag{12.49}$$

$$*b|_{S^1\times\Sigma} = 0, \tag{12.50}$$

obtained from (12.10)-(12.12) for (b,ϕ,ξ). Here, the nonlinear term

$$N_k(b,\phi,\xi) = \chi_k N_{(A_0,\Phi_0)}(b,\phi,\xi) + \left[\partial_t + \widetilde{\mathcal{H}}_0, \chi_k\right](b,\phi,\xi) \tag{12.51}$$

is also just a quadratic nonlinearity, since the commutator $\left[\partial_t + \widetilde{\mathcal{H}}_0, \chi_k\right]$ is just multiplication by a smooth function. We can regard $I_k \times Y \subseteq S^1 \times Y$ and so we can proceed as before. Here, we use the important fact that if $z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p}(S^1,\mathfrak{L})$, then $\chi_k z \in T_{\gamma_0}\mathrm{Maps}^{1-1/p}(S^1,\mathfrak{L})$ as well, since $T_{\gamma_0}\mathrm{Maps}^{1-1/p}(S^1,\mathfrak{L})$, being a path of tangent spaces, is invariant under multiplication by a function of time. Thus, the steps involving the projection Π in Step Two of the proof of Theorem 12.1 work as before, since $\chi_k z \in \ker\Pi$. When we do Step Three, we only work on a domain where $\chi_k \equiv 1$, so that the Lagrangian boundary condition is still preserved. In this way, Theorem 12.1 yields for us smoothness for $\chi_k(b,\phi,\xi)$ for all k. We can arrange the X_k and χ_k such that the union of all the intervals I'_k on which $\chi_k \equiv 1$ covers all of $\mathbb{R}$. This proves (b,ϕ,ξ) is smooth on $\mathbb{R}\times Y$, and hence so is (A,Φ). □

Next, we prove the analog of Theorem B in the periodic setting:

Theorem 12.2 *Let $p > 4$ and let $(A_i,\Phi_i) \in H^{1,p}\mathfrak{C}(S^1\times Y)$ be a sequence of solutions to (12.1)-(12.2), where $\mathfrak{L}$ is a fully gauge invariant monopole Lagrangian. Suppose we have uniform bounds*

$$\|F_{A_i}\|_{L^p(S^1\times Y)}, \|\nabla_{A_i}\Phi_i\|_{L^p(S^1\times Y)}, \|\Phi_i\|_{L^p(S^1\times Y)} \le C \tag{12.52}$$

for some constant C. Then there exists a subsequence of configurations, again denoted by (A_i,Φ_i), and a sequence of gauge transformations $g_i \in H^{2,p}\mathcal{G}(S^1\times Y)$ such that $g_i^*(A_i,\Phi_i)$ converges uniformly in $C^\infty(S^1\times Y)$.

Proof Fix any smooth reference connection A_{ref} and redefine the (A_i,Φ_i) by gauge transformations g_i that place A_i in Coulomb-Neumann gauge with respect to A_{ref}. The elliptic estimate for $d+d^*$ on 1-forms with Neumann boundary condition implies that

$$\|A_i - A_{\mathrm{ref}}\|_{H^{1,p}} \le c(\|F_{A_i}\|_{L^p} + \|(A_i - A_{\mathrm{ref}})^h\|_{L^p}), \tag{12.53}$$

where $(A_i - A_{\mathrm{ref}})^h$ is the the orthogonal projection of $(A_i - A_{\mathrm{ref}})$ onto the finite dimensional subspace

$$\{a \in \Omega^1(S^1 \times Y; i\mathbb{R}) : da = d^*a = 0, *a|_{S^1\times\Sigma} = 0\} \cong H^1(Y; i\mathbb{R}). \tag{12.54}$$

The above isomorphism is by the usual Hodge theory on manifolds with boundary. From just the bounds (12.52), we have no a priori control of $\|(A_i - A_{\mathrm{ref}})^h\|_{L^p}$. However, we still have some gauge freedom left, namely, we can consider the following group of harmonic gauge transformations

$$\mathcal{G}_{h,n} := \{g \in \mathcal{G}(S^1 \times Y) : d^*(g^{-1}dg) = 0, *dg|_{S^1\times Y} = 0\} \tag{12.55}$$

which preserve the Coulomb-Neumann gauge. The map $g \mapsto g^{-1}dg$ maps $\mathcal{G}_{h,n}$ onto the lattice $H^1(Y; 2\pi i\mathbb{Z})$ inside $H^1(Y; i\mathbb{R})$. Thus, modulo gauge transformations in $\mathcal{G}_{h,n}$, the term $(A_i - A_{\mathrm{ref}})^h$ is controlled up to a compact torus. Hence, by redefining the A_i by gauge transformations in $\mathcal{G}_{h,n}$, we can arrange that the $(A_i - A_{\mathrm{ref}})^h$ are bounded uniformly, which together with (12.52) and (12.53) implies that we have a uniform bound

$$\|A_i - A_{\mathrm{ref}}\|_{H^{1,p}} \leq C \tag{12.56}$$

for some absolute constant C (where C denotes some constant independent of the (A_i, Φ_i), whose value may change from line to line).

From (12.52) and (12.56), we have the control

$$\begin{aligned} \|\nabla_{A_{\mathrm{ref}}}\Phi_i\|_{L^p} &\leq \|\nabla_{A_i}\Phi_i\|_{L^p} + \|\rho(A_i - A_{\mathrm{ref}})\Phi_i\|_{L^p} \\ &\leq \|\nabla_{A_i}\Phi_i\|_{L^p} + \|\rho(A_i - A_{\mathrm{ref}})\|_{L^\infty}\|\Phi_i\|_{L^p} \\ &\leq C, \end{aligned} \tag{12.57}$$

due to the embedding $H^{1,p}(S^1 \times Y) \hookrightarrow L^\infty(S^1 \times Y)$ for $p > 4$. The uniform bound (12.57) and the uniform bound on $\|\Phi_i\|_{L^p}$ shows that we have the uniform bound

$$\|\Phi_i\|_{H^{1,p}} \leq C. \tag{12.58}$$

Thus, the configuration (A_i, Φ_i) is uniformly bounded in $H^{1,p}(S^1 \times Y)$. Moreover, the (A_i, Φ_i) are smooth since they solve (12.1)-(12.2) and A_i is in Coulomb-Neumann gauge with respect to a smooth connection. If we can show that the (A_i, Φ_i) are also uniformly bounded in $H^{k,2}(S^1 \times Y)$ for each $k \geq 2$, then we will be done, due to the compact embedding $H^{k+1,2}(S^1 \times Y) \hookrightarrow H^{k,2}(S^1 \times Y)$ for all $k \geq 1$ and a diagonalization argument.

Since the (A_i, Φ_i) are uniformly bounded in $H^{1,p}(S^1 \times Y)$, a subsequence converges strongly in $H^{s,p}(S^1 \times Y)$ for any $s = 1 - \epsilon$ with $\epsilon > 0$ arbitrarily small. The limiting configuration (A_∞, Φ_∞), being a weak $H^{1,p}(S^1 \times Y)$ limit of the (A_i, Φ_i), belongs to $H^{1,p}(S^1 \times Y)$, and it solves (9.7), since the equations are preserved under weak limits. In the interior, this is easy to see; on the boundary, we use the fact that $\mathrm{Maps}^{s-1/p,p}(S^1, \mathfrak{L})$ is a manifold, so that the Lagrangian boundary condition is preserved under weak limits. Since Coulomb-Neumann gauge is also preserve under weak limits, then from Theorem 12.1, we know that (A_∞, Φ_∞) is smooth.

We now apply (12.47) with (A_0, Φ_0) replaced with the smooth configuration (A_∞, Φ_∞) and (A, Φ) replaced by the (A_i, Φ_i), for large i. We can do this because the following are true: first, the (A_i, Φ_i) converge strongly to (A_∞, Φ_∞) in $H^{s,p}(S^1 \times Y)$, and $s > 1/2 + 2/p$; second, the proof of (12.47) shows that if (A_0, Φ_0) is any smooth configuration, then (12.47) holds for all (A, Φ) solving (12.1)–(12.2) sufficiently $H^{s,p}(S^1 \times Y)$ close to (A_0, Φ_0). It now follows that a subsequence of the (A_i, Φ_i) converges to (A_∞, Φ_∞) in C^∞. □

Proof of Theorem B: This follows from exhausting $\mathbb{R} \times Y$ by a sequence of compact manifolds with boundary, applying Theorem 12.2 on each of these manifolds, and a standard patching argument. See, e.g., [53, Proposition 7.6]. □

Part IV

Tools From Analysis

13 Function Spaces

In this section, we define the various function spaces needed for our analysis. We establish enough of their properties so that we may apply them in the context of elliptic boundary value problems and nonlinear partial differential equations.

13.1 The Classical Function Spaces

We define the classical Sobolev, Bessel potential, and Besov spaces. These spaces along with their basic properties are well documented, e.g., see [12], [50], and [51]. The proofs of all the statements here can be found in those references.

3.1.1 Function Spaces on $\mathbb{R}^n$

We begin by defining our spaces on $\mathbb{R}^n$ with coordinates x_j, $1 \leq j \leq n$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decaying Schwartz functions and let $\mathcal{S}'(\mathbb{R}^n)$ be its dual space, the space of tempered distributions. Given $f \in \mathcal{S}(\mathbb{R}^n)$, we have the Fourier transform

$$\mathcal{F}f(\xi) = \int e^{i\xi\cdot x} f(x)dx.$$

The Fourier transform extends to $\mathcal{S}'(\mathbb{R}^n)$ by duality. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ of nonnegative integers, we let $D^\alpha f = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} f$ be the corresponding partial derivatives of f in the sense of distributions.

Next, we consider a dyadic partition of unity as follows. Let $\psi(\xi)$ be a smooth bump function, $0 \leq \psi(\xi) \leq 1$, with $\psi(\xi)$ equal to 1 on $|\xi| \leq 1$ and ψ identically zero on $|\xi| \geq 2$. Let

$$\begin{aligned} \varphi_0(\xi) &= \psi(\xi) \\ \varphi_j(\xi) &= \psi(2^{-j}\xi) - \psi(2^{-j-1}\xi), \quad j \geq 1. \end{aligned}$$

Then we have $\sum_{j=0}^{\infty} \varphi_j(\xi) \equiv 1$ with $\operatorname{supp} \varphi_j \subset [2^{j-1}, 2^{j+1}]$ for $j \geq 1$.

Given a tempered distribution f, we let

$$f_j = \mathcal{F}^{-1}\varphi_j \mathcal{F} f$$

be its jth dyadic component. The decomposition of f into its dyadic components $\{f_j\}_{j=0}^\infty$ is known as the Littlewood-Paley decomposition.

On $\mathbb{R}^n$, let $L^p(\mathbb{R}^n)$ and $C^\alpha(\mathbb{R}^n)$ denote the usual Lebesgue and Holder spaces of order p and α, respectively, where $1 \le p \le \infty$ and $\alpha \ge 0$. In addition to these, we have the following classical function spaces:

Definition 13.1 (i) For $s \in \mathbb{Z}_+$ a nonnegative integer and $1 \le p \le \infty$, define the *Sobolev spaces*

$$W^{s,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \quad \|f\|_{W^{s,p}} = (\sum_{|\alpha| \le s} \|D^\alpha f\|_{L^p}^p)^{1/p} < \infty\}, \qquad p < \infty \tag{13.1}$$

$$W^{s,\infty}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{W^{s,\infty}} = \sup_{|\alpha| \le s} \|D^\alpha f\|_{L^\infty} < \infty\}. \tag{13.2}$$

(ii) For $s \in \mathbb{R}$ and $1 < p < \infty$, define the *Bessel potential spaces*

$$H^{s,p}(\mathbb{R}^n) := \Big\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^{s,p}} = \Big\|\Big(\sum_{j=0}^\infty |2^{sj} f_j|^2\Big)^{1/2}\Big\|_{L^p} < \infty\Big\}. \tag{13.3}$$

(iii) For $s \in \mathbb{R}$, $1 < p < \infty$, define the *Besov spaces*[1]

$$B^{s,p}(\mathbb{R}^n) = \Big\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^{s,p}} = \Big\|\Big(\sum_{j=0}^\infty |2^{sj} f_j|^p\Big)^{1/p}\Big\|_{L^p} < \infty\Big\}. \tag{13.4}$$

(iv) Define $A^{s,p}$ to be shorthand for either $H^{s,p}$ or $B^{s,p}$. The spaces $A^{s,p}$ are also a special case of what are known as *Triebel-Lizorkin spaces.*

Of the above Banach spaces, the Sobolev spaces $W^{s,p}$ are the ones most naturally occurring for many of the basic problems in analysis. The Bessel potential spaces $H^{s,p}$ arise from (complex) interpolation between the Sobolev spaces, where we may think of $f \in H^{s,p}$ as having s derivatives in L^p. This is most clearly illustrated when $p = 2$, where then $H^{s,p}$ is usually just denoted as H^s. For general p, we have the following result:

Theorem 13.2 *[50, Theorem 2.3.3] For $1 < p < \infty$, $H^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ for s a non-negative integer.*

Indeed, when $s = 0$, then Theorem 13.2 tells us that

$$\Big\|\Big(\sum_{j=0}^\infty |f_j|^2\Big)^{1/2}\Big\|_{L^p} \sim \|f\|_{L^p}.$$

[1] The classical Besov spaces are usually denoted with two parameters $B^s_{p,q}$. We take $p = q$. There are also many other equivalent norms that can be used to define the Besov spaces. Our choice of norm reflects their similarity with $H^{s,p}$.

This is the classical Littlewood-Paley Theorem.

The Besov spaces naturally arise because they are the boundary values of Sobolev spaces. More precisely, let $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ be the hyperplane $x_n = 0$. Given a fixed $m \in \mathbb{Z}_+$ and f a function on $\mathbb{R}^n$, let

$$r_m f = \left(f|_{\mathbb{R}^{n-1}}, \partial_{x_n} f|_{\mathbb{R}^{n-1}}, \ldots, \partial_{x_n}^m f|_{\mathbb{R}^{n-1}}\right) \tag{13.5}$$

be the trace of f of order m along the hyperplane $\mathbb{R}^{n-1}$. We have the following theorem:

Theorem 13.3 *(i) For $s > m+1/p$ and $m \in \mathbb{Z}_+$, the trace map r_m extends to a bounded operator*

$$r_m : H^{s,p}(\mathbb{R}^n) \to \oplus_{j=0}^{m-1} B^{s-1/p-j}(\mathbb{R}^{n-1}) \tag{13.6}$$

(ii) For any $s \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, there exists an extension map $e_m : \oplus_{j=0}^{m-1} B^{s-1/p-j,p}(\mathbb{R}^{n-1}) \to H^{s,p}(\mathbb{R}^n)$ such that for $s > m + 1/p$, we have $r_m e_m = \mathrm{id}$.

(iii) $H^{s,p}(\mathbb{R}^n)$ may be replaced with $B^{s,p}(\mathbb{R}^n)$ in the above.

When $p = 2$, we have

$$B^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n) \tag{13.7}$$

for all s, and so the above theorem is a generalization of the fact that the trace of an element of $H^s(\mathbb{R}^n)$ lies in $H^{s-1/2}(\mathbb{R}^n)$ for $s > 1/2$. Furthermore, because $\ell^p \subseteq \ell^q$ whenever $p \geq q$, we have the trivial inclusions

$$\begin{aligned} B^{s,p}(\mathbb{R}^n) &\subseteq H^{s,p}(\mathbb{R}^n) \qquad p \leq 2 \\ H^{s,p}(\mathbb{R}^n) &\subseteq B^{s,p}(\mathbb{R}^n) \qquad p \geq 2. \end{aligned}$$

For $s > 0$, we can also write the Besov space norm in terms of finite differences in space rather than in terms of the Littlewood-Paley decomposition in frequency space. For any $h \in \mathbb{R}^n$, define the operator

$$\delta_h f = f(x+h) - f(x).$$

Using this operator, we have the following proposition:

Proposition 13.4 *For $s > 0$ and $1 < p < \infty$, let m be any integer such that $m > s$. Then an equivalent norm for $B^{s,p}(\mathbb{R}^n)$ is given by*

$$\|f\|_{B^{s,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n}^{\infty} \left\||h|^{-s}\delta_h^m f\right\|_{L^p(\mathbb{R}^n)}^p \frac{1}{|h|^n} dh\right)^{1/p}. \tag{13.8}$$

Remark 13.5 The spaces $H^{s,p}(\mathbb{R}^n)$ and $B^{s,p}(\mathbb{R}^n)$ satisfy

$$H^{s_1,p}(\mathbb{R}^n) \subseteq B^{s_2,p}(\mathbb{R}^n) \subseteq H^{s_3,p}(\mathbb{R}^n)$$

for all $s_1 > s_2 > s_3$, for $1 < p < \infty$. This is a simple consequence of the definitions (13.3) and (13.4). Thus, we see that the most important features of the $B^{s,p}$ and $H^{s,p}$ spaces are determined by the exponents s, p, with the distinction between the Besov and Bessel potential topologies for fixed s and p being a more refined property. In this sense, for most purposes, the spaces $B^{s,p}$ and $H^{s,p}$ are "nearly identical", and many results concerning one of these spaces implies the same result for the other. This is why we adopt the common notation of using $A^{s,p}$ to denote either $H^{s,p}$ or $B^{s,p}$. Whenever, $A^{s,p}$ appears in multiple instances in a statement or formula, we always mean that all instances of $A^{s,p}$ are either $H^{s,p}$ or $B^{s,p}$.

We have the following fundamental properties:

Proposition 13.6 *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then the space of compactly supported functions $C_0^\infty(\mathbb{R}^n)$ is dense in $A^{s,p}(\mathbb{R}^n)$. Moreover, $A^{-s,p'}(\mathbb{R}^n)$ is the dual space of $A^{s,p}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$.*

Proposition 13.7 *(Lift Property) Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then*

$$A^{s,p}(\mathbb{R}^n) = \{f \in A^{s-1,p}(\mathbb{R}^n) : \frac{\partial f}{\partial x^i} \in A^{s-1,p}(\mathbb{R}^n), 1 \leq i \leq n\}.$$

3.1.1 Function Spaces on an Open Subset of $\mathbb{R}^n$

Let Ω be an open subset of $\mathbb{R}^n$. Unless otherwise stated, we assume for simplicity that Ω is bounded and has smooth boundary, though many of the results that follow carry over for more general open sets. Given any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, we can consider its restriction $r_\Omega(f)$ to $(C_0^\infty(\Omega))'$. Then we have the corresponding function spaces on Ω:

Definition 13.8 For $s \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$, the space $W^{s,p}(\Omega)$ is the space of restrictions to Ω of elements of $W^{s,p}(\mathbb{R}^n)$, where the norm on $W^{s,p}(\Omega)$ is given by

$$\|f\|_{W^{s,p}(\Omega)} = \inf_{g: r_\Omega(g) = f} \|g\|_{W^{s,p}(\mathbb{R}^n)}.$$

For $s \in \mathbb{R}$ and $1 < p < \infty$, the spaces $H^{s,p}(\Omega)$ and $B^{s,p}(\Omega)$ are defined similarly.

If we consider the function space

$$\tilde{A}^{s,p}(\Omega) := \{f \in A^{s,p}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\},$$

then an equivalent definition of $A^{s,p}(\Omega)$ is

$$A^{s,p}(\Omega) = A^{s,p}(\mathbb{R}^n)/\tilde{A}^{s,p}(\mathbb{R}^n \setminus \bar{\Omega}). \tag{13.9}$$

Furthermore, we have the following:

Proposition 13.9 *Let $-\infty < s < \infty$ and $1 < p < \infty$. Then $C_0^\infty(\Omega)$ is dense in $\tilde{A}^{s,p}(\Omega)$. Moreover, $A^{-s,p'}(\Omega)$ is the dual space of $\tilde{A}^{s,p}(\Omega)$, where $1/p + 1/p' = 1$.*

Define the upper half-space

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

We have the following extension property:

Theorem 13.10 *Let $1 < p < \infty$. For any $k \in \mathbb{N}$, there exists an extension operator*

$$E_k : A^{s,p}(\mathbb{R}^n_+) \to A^{s,p}(\mathbb{R}^n)$$

for $|s| < k$.

3.1.1 Function Spaces on Manifolds

Ultimately, the function spaces which are important for us are those which are defined on manifolds (with and without boundary). Let X be a compact n-manifold or an open subset of it. We can assign to X the data of an atlas $\{(U_i, \varphi_i, \Phi_i)\}$, where: (1) the U_i are a finite open cover of X; (2) the φ_i are a partition of unity with $\operatorname{supp}\varphi_i \subset U_i$; (3) each Φ_i is a map from U_i to $\mathbb{R}^n$, where Φ_i is a diffeomorphism onto an open subset of $\mathbb{R}^n$ if $\bar{U}_i \subset \mathring{X}$ or otherwise, Φ_i is a diffeomorphism onto an open subset of $\overline{\mathbb{R}^n_+}$ with $\Phi_i(U_i \cap \partial X) \subset \partial\overline{\mathbb{R}^n_+}$. With this data, we can define the function spaces $A^{s,p}(X)$ in terms of the function spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Definition 13.11 Let X be a compact manifold or an open subset of it. Let $\{(U_i, \varphi_i, \Phi_i)\}$ be an atlas as above. Then for $-\infty < s < \infty$ and $1 < p < \infty$, we define $A^{s,p}(X)$ to be those distributions f on X such that

$$\|f\|_{A^{s,p}(X)} = \left(\sum_{U_i \subset \mathring{X}} \|\Phi_i^*(\varphi_i f)\|^p_{A^{s,p}(\mathbb{R}^n)} + \sum_{U_i \cap \partial X \neq \emptyset} \|\Phi_i^*(\varphi_i f)\|^p_{A^{s,p}(\mathbb{R}^n_+)} \right)^{1/p} < \infty.$$

We define $W^{s,p}(X)$ for $s \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$ similarly.

If we have two different atlases, the following proposition implies that we obtain equivalent norms:

Proposition 13.12 *Let $f \in A^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$ and $1 < p < \infty$. (i) If $\varphi \in C_0^\infty(\mathbb{R}^n)$ then $\varphi f \in A^{s,p}(\mathbb{R}^n)$. (ii) If Φ is a diffeomorphism of $\mathbb{R}^n$ which is equal to the identity outside a compact set, then $\Phi^*(f) \in A^{s,p}(\mathbb{R}^n)$.*

In particular, if X is a bounded open subset of $\mathbb{R}^n$, the above furnishes a definition of $A^{s,p}(X)$. On the other hand, we also defined $A^{s,p}(X)$ to be the restrictions to X of $A^{s,p}(\mathbb{R}^n)$. These two definitions of $A^{s,p}(X)$ yield equivalent norms. Consequently, if X is a compact manifold and $\tilde{X}$ is a closed manifold containing X, we have the following:

Proposition 13.13 *For $-\infty < s < \infty$ and $1 < p < \infty$, $A^{s,p}(X)$ is the space of restrictions to X of $A^{s,p}(\tilde{X})$.*

Corollary 13.14 *Let X be a compact manifold (with or without boundary) or Euclidean space. If D is a differential operator of order m, then $D : A^{s,p}(X) \to A^{s-m,p}(X)$ for all $s \in \mathbb{R}$ and $1 < p < \infty$.*

Because function spaces defined on manifolds are locally the function spaces defined on Euclidean space, many of the properties of the latter carry over to the manifold case. For instance, if $\tilde{X}$ is any closed manifold containing the manifold X, we can define

$$\tilde{A}^{s,p}(X) = \{f \in A^{s,p}(\tilde{X}) : \operatorname{supp} f \subseteq X\}. \tag{13.10}$$

We have the following theorem:

Theorem 13.15 *Let X be a compact manifold. We have that $C^\infty(X)$ is dense in $A^{s,p}(X)$ and multiplication by a smooth function defines a bounded operator. Moreover, for any $s \in \mathbb{R}$, $A^{-s,p'}(X)$ is the dual space of $\tilde{A}^{s,p}(X)$, where $p' = p/(p-1)$. If X is closed or $s < 1/p$, then $\tilde{A}^{s,p}(X) = A^{s,p}(X)$.*

The trace theorem, Theorem 13.3, readily generalizes to manifolds with boundary:

Theorem 13.16 *Let X be a compact manifold with boundary ∂X.*

(i) For $s > m + 1/p$ and $m \in \mathbb{Z}_+$, the trace map (13.5) extends to a bounded operator

$$r_m : H^{s,p}(X) \to \oplus_{j=0}^{m-1} B^{s-1/p-j}(\partial X). \tag{13.11}$$

(ii) For any $s \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, there exists an extension map $e_m : \oplus_{j=0}^{m-1} B^{s-1/p-j,p}(\partial X) \to H^{s,p}(X)$ such that for $s > m + 1/p$, we have $r_m e_m = \mathrm{id}$.

(iii) $H^{s,p}(X)$ may be replaced with $B^{s,p}(X)$ in the above.

Further Properties

In the following, X is a compact manifold (with or without boundary).

Theorem 13.17 *(Embedding Theorem) Let $-\infty < t < s < \infty$ and $1 < p, q < \infty$ with*

$$s - n/p \geq t - n/q \tag{13.12}$$

(i) We have embeddings

$$B^{s,p}(X) \hookrightarrow B^{t,q}(X) \cap H^{t,q}(X) \tag{13.13}$$

$$H^{s,p}(X) \hookrightarrow H^{t,q}(X) \cap B^{t,q}(X). \tag{13.14}$$

If the inequality (13.12) is strict, these embeddings are compact.

(ii) We have the monotonicity property

$$H^{s,p}(X) \subseteq H^{s,q}(X), \qquad p > q. \tag{13.15}$$

(iii) If $t > 0$ is not an integer, then

$$H^{n/p+t,p}(X) \hookrightarrow C^t(X)$$
$$B^{n/p+t,p}(X) \hookrightarrow C^t(X).$$

Next, we have a multiplication theorem. Namely, given two functions f and g, we wish to know in which space their product fg lies (where it is assumed that f and g are sufficiently regular so that their product makes sense as a distribution).

Theorem 13.18 *(Multiplication Theorem)*

(i) For all $s > 0$, we have $A^{s,p}(X) \cap L^\infty(X)$ is an algebra. Moreover, we have the estimate

$$\|fg\|_{A^{s,p}} \leq C(\|f\|_{A^{s,p}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{A^{s,p}}).$$

In particular, if $s > n/p$, then $A^{s,p}(X)$ is an algebra.

(ii) Let $s_1 \leq s_2$ and suppose $s_1 + s_2 > n\max(0, \frac{2}{p} - 1)$. Then we have a continuous multiplication map

$$A^{s_1,p}(X) \times A^{s_2,p}(X) \to A^{s_3,p}(X),$$

where

$$s_3 = \begin{cases} s_1 & \text{if } s_2 > n/p \\ s_1 + s_2 - n/p & \text{if } s_2 < n/p. \end{cases}$$

Both statements are standard facts, whose proofs involve the paraproduct calculus. For (i), see e.g. [49]. For (ii), see [41]. □

Theorem 13.19 *(Fubini Property) For any $s > 0$, we have*

$$B^{s,p}(X_1 \times X_2) = L^p(X_1, B^{s,p}(X_2)) \cap L^p(X_2, B^{s,p}(X_1)).$$

Proof The case when X_1 and X_2 are closed manifolds follows from the Euclidean case, which is proved in [51, Theorem 2.5.13]. Now suppose X_1 or X_2 has boundary, say both. Let $\tilde{X}_1$ and $\tilde{X}_2$ be closed manifolds extending X_1 and X_2, respectively. Then on the one hand, by definition, we can find a function $\tilde{f}$ on $\tilde{X}_1 \times \tilde{X}_2$ such that $\tilde{f}|_{X_1 \times X_2} = f$ and

$$\|\tilde{f}\|_{B^{(s_1,s_2),p}(\tilde{X}_1 \times \tilde{X}_2)} \leq 2\|f\|_{B^{(s_1,s_2),p}(X_1 \times X_2)}.$$

The Fubini property for $\tilde{f}$ on $\tilde{X}_1 \times \tilde{X}_2$, i.e. $\tilde{f} \in L^p(\tilde{X}_1, B^{s,p}(\tilde{X}_2)) \cap L^p(\tilde{X}_2, B^{s,p}(\tilde{X}_1))$ implies via restriction that $f \in L^p(X_1, B^{s,p}(X_2)) \cap L^p(X_2, B^{s,p}(X_1))$. Conversely, for any fixed s, we know that there exists bounded extension maps $E_i : C^\infty(X_i) \to C^\infty(\tilde{X}_i)$ such that

$$\begin{aligned} E_i &: B^{s,p}(X_i) \to B^{s,p}(\tilde{X}_i) \\ E_i &: L^p(X_i) \to L^p(\tilde{X}_i), \qquad i = 1, 2. \end{aligned}$$

We can compose the extensions E_1 and E_2 to give us an extension map on the product:

$$E = E_1 \circ E_2 = E_2 \circ E_1 : C^\infty(X_1 \times X_2) \to C^\infty(\tilde{X}_1 \times \tilde{X}_2).$$

Thus, given $f \in L^p(X_1, B^{s,p}(X_2)) \cap L^p(X_2, B^{s,p}(X_1))$, we have that

$$E(f) \in L^p(\tilde{X}_1, B^{s,p}(\tilde{X}_2)) \cap L^p(\tilde{X}_2, B^{s,p}(\tilde{X}_1)) = B^{s,p}(\tilde{X}_1 \times \tilde{X}_2)$$

. Restricting back to $X_1 \times X_2$, we conclude that $f \in B^{s,p}(X_1 \times X_2)$. □

13.2 Anisotropic Function Spaces

In the previous section, we defined the classical function spaces, all of which were isotropic. That is, the regularity parameter s measures smoothness in all directions equally. On the other hand, it is natural to consider spaces that measure different amounts of regularity in different directions. More precisely, suppose we are given a splitting of $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The anisotropic function spaces we consider are those that possess an extra degree of regularity in the second space factor $\mathbb{R}^{n_2}$. To measure this, we introduce the following family of Bessel potential operators acting on $\mathbb{R}^{n_2}$:

$$J^{s_2}_{(2)} f = \mathcal{F}^{-1} \langle \xi^{(2)} \rangle^{s_2} \mathcal{F} f, \quad s_2 \in \mathbb{R}, \tag{13.16}$$

where $\langle \xi^{(2)} \rangle := (1 + |\xi^{(2)}|^2)^{1/2}$ for $\xi^{(2)} \in \mathbb{R}^{n_2}$. These operators are the anisotropic versions of the usual Bessel potential operators acting on $\mathbb{R}^n$:

$$J^s f = \mathcal{F}^{-1} \langle \xi \rangle^s \mathcal{F} f. \tag{13.17}$$

We now define the following anisotropic function spaces:

Definition 13.20 Let $s_1 \in \mathbb{R}$, $s_2 \geq 0$, and $1 < p < \infty$.

(i) Define the *anisotropic Bessel potential spaces*

$$H^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in S'(\mathbb{R}^n) : \|f\|_{H^{(s_1,s_2),p}} = \|J^{s_2}_{(2)} f\|_{H^{s_1,p}} < \infty\}. \tag{13.18}$$

(ii) Define the *anisotropic Besov spaces*

$$B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in S'(\mathbb{R}^n) : \|f\|_{B^{(s_1,s_2),p}} = \|J^{s_2}_{(2)} f\|_{B^{s_1,p}} < \infty\}. \tag{13.19}$$

Thus, the s_1 parameter is an isotropic parameter which measures how much regularity a function has in all directions, while the s_2 is the anisotropic parameter which measures extra smoothness in the $\mathbb{R}^{n_2}$ directions. As a special case, when s_1 and s_2 are nonnegative integers, we see that $H^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are anisotropic Sobolev spaces.

We can also define anisotropic spaces on products of open subsets of Euclidean space in the same way as we did for isotropic function spaces. Thus, if $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$ are bounded open subsets (with smooth boundary), we define $\|f\|_{B^{(s_1,s_2),p}(\Omega_1 \times \Omega_2)}$ to be the closure of all smooth functions on $\Omega_1 \times \Omega_2$ in the norm

$$\|f\|_{B^{(s_1,s_2),p}(\Omega_1 \times \Omega_2)} := \inf_{\substack{g \in B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \\ g|_{\Omega_1 \times \Omega_2} = f}} \|g\|_{B^{(s_1,s_2),p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \tag{13.20}$$

From this, using local product coordinate charts, we can define anisotropic Besov spaces on products of manifolds with boundary. More precisely, we can define the spaces $B^{(s_1,s_2),p}(X_1\times X_2)$, where X_1 and X_2 are either Euclidean space or compact manifolds. In detail, let X be any compact n-manifold (with or without boundary). We can assign to X the data of a finite collection of triples $\{(U_i,\varphi_i,\Phi_i)\}$, which we call a *coordinate system*, where: (1) the U_i are a finite open cover of X; (2) the φ_i are a partition of unity with $\operatorname{supp}\varphi_i\subset U_i$; (3) each Φ_i is a map from U_i to $\mathbb{R}^n$, where Φ_i is a diffeomorphism onto an open subset of $\mathbb{R}^n$ if $\bar{U}_i\subset\mathring{X}$ or otherwise, Φ_i is a diffeomorphism onto an open subset of $\overline{\mathbb{R}^n_+}$ with $\Phi_i(U_i\cap\partial X)\subset\partial\overline{\mathbb{R}^n_+}$. On Euclidean space, we may also assign a coordinate system, namely the trivial atlas $\{(X,1,\mathrm{id})\}$.

Definition 13.21 Let X_1 and X_2 be either Euclidean space or compact manifolds and let $\{U_i^{(j)},\varphi_i^{(j)},\Phi_i^{(j)}\}$ be coordinate systems for X_j, $j=1,2$ as above. Let $s_1\in\mathbb{R}$, $s_2\geq 0$, and $1<p<\infty$.

(i) Define $B^{(s_1,s_2),p}(X_1\times X_2)$ to be the closure of the space of smooth compactly supported functions on $X_1\times X_2$ under the norm

$$\|f\|_{B^{(s_1,s_2),p}(X_1\times X_2)}=\left(\sum_{i,i'}\|(\Phi_i^{(1)}\times\Phi_{i'}^{(2)})^*\varphi_i^{(1)}\varphi_{i'}^{(2)}f\|^p_{B^{(s_1,s_2),p}(\mathbb{R}^{n_1}_*\times\mathbb{R}^{n_2}_*)}\right)^{1/p},$$

where $\mathbb{R}^{n_j}_*$ stands for $\mathbb{R}^{n_j}$ or $\mathbb{R}^{n_j}_+$, accordingly, $j=1,2$.

(ii) Suppose X_1 is a bounded open subset of $\mathbb{R}^n$ and X_2 is a compact manifold. Then an equivalent definition for $B^{(s_1,s_2),p}(X_1\times X_2)$ is that it is the space of restrictions of $B^{(s_1,s_2),p}(\mathbb{R}^n\times X_2)$, i.e., the norm on $B^{(s_1,s_2),p}(X_1\times X_2)$ is given by

$$\|f\|_{B^{(s_1,s_2),p}(X_1\times X_2)}=\inf_{\substack{g\in B^{(s_1,s_2),p}(\mathbb{R}^n\times X_2)\\ g|_{X_1\times X_2}=f}}\|g\|_{B^{(s_1,s_2),p}(\mathbb{R}^n\times X_2)}. \tag{13.21}$$

We have that multiplication by a smooth compactly supported function is bounded on $B^{(s_1,s_2),p}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ (see [36]). Likewise, if Φ_j are diffeomorphisms of $\mathbb{R}^{n_j}$ that are the identity outside a compact set, then pullback by $\Phi_1\times\Phi_2$ is a bounded operator on $B^{(s_1,s_2),p}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. It follows from this that different choices of atlases define equivalent norms in the above definition.

Note that if X_1 and X_2 are manifolds with boundary, then $X_1\times X_2$ is a manifold with corners. Nevertheless, the above definitions still make sense and the corresponding anisotropic function spaces are still well-behaved, as we will see below.

Further reading about the anisotropic function spaces we have defined can be found in [36]. We state the main results we need from [36], which are essentially the generalizations of the basic properties of isotropic function spaces from the previous section to the anisotropic case.

The first result we have is the generalization of trace and extension properties in Theorem 13.16. For simplicity, we state this generalization for the case $m=1$. Recall that if X is a compact manifold with boundary, then we have a (zeroth order) trace map r :

$B^{s,p}(X) \to B^{s-1/p,p}(\partial X)$, $s > 1/p$, which "costs" us $1/p$ derivatives and an extension map $B^{s-1/p,p}(\partial X) \to B^{s,p}(X)$ which gains us $1/p$ derivatives. Suppose now we have $X = X_1 \times X_2$, where X_1 and X_2 are compact manifolds (with or without boundary). When taking a trace to the boundary, the anisotropy of a function can be either tangential or normal to the boundary. When the anisotropy of a function is tangential to the boundary, then the trace and extension operators preserve this anisotropy, since tangential operations commute with such operators. On the other hand, if there is anisotropy in the normal direction, then if there is enough anisotropy, taking a trace costs us $1/p$ derivatives only in the anisotropic directions, when $p \geq 2$. This is summarized in the following:

Theorem 13.22 *(Anisotropic Traces and Extensions) Let $X = X_1 \times X_2$.*

(i) (Tangential anisotropy) Suppose X_1 has boundary ∂X_1 and X_2 is closed. Then for $s_1 > 1/p$ and $s_2 \geq 0$, the trace map satisfies

$$r : A^{(s_1,s_2),p}(X_1 \times X_2) \to B^{(s_1-1/p,s_2),p}(\partial X_1 \times X_2). \tag{13.22}$$

Furthermore, for all $s_1 \in \mathbb{R}$, there exists a boundary extension map

$$e : B^{(s_1-1/p,s_2),p}(\partial X_1 \times X_2) \to A^{(s_1,s_2),p}(X_1 \times X_2),$$

and for $s_1 > 1/p$, we have $re = \mathrm{id}$. Moreover, let $\tilde{X}_1$ be any closed manifold extending X_1. Then for every $k \in \mathbb{N}$, we have an extension map

$$E_k : A^{(s_1,s_2),p}(X_1 \times X_2) \to A^{(s_1,s_2),p}(\tilde{X}_1 \times X_2),$$

for $|s_1| < k$.

(ii) (Mixed anisotropy) Suppose X_2 has boundary ∂X_2 and X_1 is closed. Then for $s_1 \geq 0$, $s_2 > 1/p$, the trace map satisfies

$$r : H^{(s_1,s_2),p}(X_1 \times X_2) \to H^{(s_1,s_2-1/p-\epsilon_2),p}(X_1 \times \partial X_2), \tag{13.23}$$

$$r : B^{(s_1,s_2),p}(X_1 \times X_2) \to B^{(s_1-\epsilon_1,s_2-1/p-\epsilon_2),p}(X_1 \times \partial X_2), \tag{13.24}$$

where ϵ_1 and ϵ_2 satisfy the following:

(a) if $p > 2$, then $\epsilon_1 = 0$ and $\epsilon_2 > 0$ is arbitrary;

(b) if $p = 2$, then $\epsilon_1 = \epsilon_2 = 0$;

(c) if $1 < p < 2$, then $\epsilon_1, \epsilon_2 > 0$ are arbitrary.

Next, we recall that for isotropic Besov spaces, we have the embedding

$$B^{s,p}(X) \hookrightarrow C^0(X) \tag{13.25}$$

if $s > n/p$, where $n = \dim X$. Thus, we have the following corollary:

Corollary 13.23 *Let $X = X_1 \times X_2$ and $n_i = \dim X_2$. Then if $s_1 > n_1/p$ and $s_2 > n_2/p$, we have $B^{(s_1,s_2),p}(X_1 \times X_2) \hookrightarrow C^0(X_1 \times X_2)$.*

Proof By (13.25) and Theorem 13.22, we can take successive traces to conclude $B^{(s_1,s_2),p}(X_1 \times X_2) \hookrightarrow C^0(X_1, B^{s_1+s_2-n_1/p,p}(X_2))$ since $s_1 > n_1/p$. By (13.25), we have $B^{s_1+s_2-n_1/p,p}(X_2) \hookrightarrow C^0(X_2)$ since $s_1 + s_2 - n_1/p > n_2/p$, whence the theorem follows. □

Recall that we have a multiplication theorem for Besov spaces. Such a theorem is proved using the paraproduct calculus. By redoing the carefully paramultiplication for anisotropic Besov spaces, one can also prove an anisotropic muliplication theorem. We state such result one for $p = 2$:

Theorem 13.24 *(Anisotropic Multiplication) Let* $\dim X_i = n_i$, $i = 1, 2$ *and suppose* $s_1 > n_1/2$. *Let* $s_2', s_2'' \geq 0$ *and let* $s_2 \leq \min(s_2', s_2'')$ *satisfy* $s_2 < s_1 + s_2' + s_2'' - \frac{n_1+n_2}{2}$. *Then we have a multiplication map*

$$[B^{(s_1,s_2'),2}(X_1 \times X_2) \cap L^\infty] \times [B^{(s_1,s_2''),2}(X_1 \times X_2) \cap L^\infty] \to B^{(s_1,\max(s_2,0)),2}(X_1 \times X_2). \quad (13.26)$$

Also of fundamental importance is that pseudodifferential operators and product-type pseudodifferential operators are bounded on our anisotropic function spaces (and hence also on the classical isotropic function spaces as a special case). See Theorems 15.2 and 15.4.

13.3 Vector-valued Function Spaces

Up to now, we have considered only scalar valued functions. It is also possible to consider functions with values in a Banach space $\mathcal{X}$. In the function space literature, such functions are known more succinctly as vector-valued functions. From Definition 13.1, we can generalize the definition of the scalar-valued classical function spaces to vector-valued case in a straightforward way:

Definition 13.25 Let $\mathcal{X}$ be a Banach space.

(i) For $s \in \mathbb{Z}_+$ a nonnegative integer and $1 \leq p < \infty$, define the *vector-valued Sobolev spaces*

$$W^{s,p}(\mathbb{R}^n, \mathcal{X}) = \{f \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X}) : \quad \|f\|_{W^{s,p}(\mathbb{R}^n,\mathcal{X})} = (\sum_{|\alpha| \leq s} \|\|D^\alpha f\|_{\mathcal{X}}\|^p_{L^p(\mathbb{R}^n)})^{1/p} < \infty\}, \quad (13.27)$$

Define $W^{s,\infty}(\mathbb{R}^n, \mathcal{X})$ with the obvious modifications of the above.

(ii) For $s \in \mathbb{R}$ and $1 < p < \infty$, define the *vector-valued Bessel potential spaces*

$$H^{s,p}(\mathbb{R}^n, \mathcal{X}) := \Big\{f \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X}) : \|f\|_{H^{s,p}(\mathbb{R}^n,\mathcal{X})} = \Big\|\Big(\sum_{j=0}^\infty \|2^{sj} f_j\|^2_{\mathcal{X}}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} < \infty\Big\}. \quad (13.28)$$

(iii) For $s \in \mathbb{R}$, $1 < p < \infty$, define the *vector-valued Besov spaces*

$$B^{s,p}(\mathbb{R}^n, \mathcal{X}) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X}) : \|f\|_{B^{s,p}(\mathbb{R}^n,\mathcal{X})} = \Big\| \Big(\sum_{j=0}^{\infty} \|2^{sj} f_j\|_{\mathcal{X}}^p \Big)^{1/p} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}. \tag{13.29}$$

Let us clarify some of the objects occurring in the above definitions. The space $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ is the space of $\mathcal{X}$-valued tempered distributions, i.e. the space of continuous maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{X}$. When $s = 0$, the space $W^{s,p}(\mathbb{R}^n, \mathcal{X}) = L^p(\mathbb{R}^n, \mathcal{X})$ is also referred to as a *Bochner space*. Here, the f_j are the dyadic Littlewood-Paley components of f, defined just as in the scalar case.

When $\mathcal{X}$ is a Hilbert space, the Fourier analytic techniques of the scalar-valued function spaces follow through for vector-valued spaces. For example, one has the following *operator*-valued multiplier result, which generalizes the classical (scalar) Mihklin multiplier theorem:

Theorem 13.26 *[15] Let $m : \mathbb{R}^n \to \mathcal{B}(\mathcal{X})$ be a map into the space of bounded operators on a Hilbert space $\mathcal{X}$. Suppose $|x|^\alpha \partial_x^\alpha m(x) \in \mathcal{B}(\mathcal{X})$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and all multi-indices α. Then*

$$f \mapsto \mathcal{F}^{-1} m \mathcal{F} f$$

is bounded on $L^p(\mathbb{R}^n, \mathcal{X})$ for all $1 < p < \infty$.

On the other hand, if $\mathcal{X}$ is a general Banach space, we no longer have a multiplier theorem as above, even for multipliers with values in scalar operators. However, when $\mathcal{X}$ is a Banach space that satisfies the so called UMD property[2], then the above result holds if $m(x)$ is a scalar operator for every x, i.e. we have a scalar operator-valued multiplier theorem for UMD Banach spaces. In particular, we have the following:

Theorem 13.27 *Let $\mathcal{X}$ be a UMD Banach space. Then $W^{s,p}(\mathbb{R}^n, \mathcal{X}) = H^{s,p}(\mathbb{R}^n, \mathcal{X})$ for all $s \in \mathbb{Z}_+$.*

If one wants to generalize Theorem 13.26 to UMD Banach spaces, one needs a stronger condition on the multiplier m, namely that it be *R-bounded*. This is a technical condition, which unfortunately, is not easy to verify (see [15]). For comparison, vector-valued Besov spaces are more well-behaved than vector-valued Lebesgue spaces with respect to the above considerations, since the scalar operator-valued multiplier theorem holds on $B^{s,p}(\mathbb{R}^n, \mathcal{X})$ for any Banach space $\mathcal{X}$, not just those that are UMD (see [2]).

From the above considerations, one sees that vector-valued function spaces need to be treated with more care than their scalar-valued counterparts. However, we should note that all the spaces $A^{s,p}(X)$, for X a Euclidean space or a compact manifold (with boundary), are in fact UMD, $1 < p < \infty$, $s \in \mathbb{R}$. Moreover, the results which we need for vector-valued function spaces hold without restriction on Banach space $\mathcal{X}$. Namely, we have the following vector-valued results which generalize the corresponding well-known scalar-valued results:

Theorem 13.28 *Let $\mathcal{X}$ be a Banach space.*

[2] A Banach space $\mathcal{X}$ has the UMD property if and only if the Hilbert transform is bounded on $L^p(\mathbb{R}, \mathcal{X})$ for some (and hence any) $1 < p < \infty$.

(i) *(Sobolev embedding) Let* $1 \leq p, q < \infty$ *and* k *a nonnegative integer. Then if* $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, *we have the embedding* $W^{k,p}(\mathbb{R}^n, \mathcal{X}) \subset L^q(\mathbb{R}^n, \mathcal{X})$. *If* $k > n/p$, *then we have the embedding* $W^{k,p}(\mathbb{R}^n, \mathcal{X}) \subset C^0(\mathbb{R}^n, \mathcal{X})$.

(ii) *(Gagliardo-Nirenberg inequality) Let* m *be a positive integer, let* $1 < p, q \leq \infty$, *and let* β *be a multi-index such that* $0 < |\beta| < m$. *For*

$$\theta = \frac{|\beta|}{m}, \qquad \frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{p},$$

there exists a constant c *such that*

$$\|D^\beta f\|_{L^r(\mathbb{R}^n,\mathcal{X})} \leq c\|f\|^{1-\theta}_{L^q(\mathbb{R}^n,\mathcal{X})} \left(\sum_{|\alpha|=m} \|D^\alpha f\|_{L^p(\mathbb{R}^n,\mathcal{X})} \right)^\theta .$$

Proof (i) The usual scalar Sobolev embedding theorem can be proven using integration by parts and applications of Holder's inequality. A proof of this can be found, e.g. in [11]. The proof there generalizes to the vector-valued case from the results of [39], which establish that if $f \in W^{1,1}(\mathbb{R}, \mathcal{X})$, then $\|f(\cdot)\|_{\mathcal{X}} \in W^{1,1}(\mathbb{R})$.

(ii) This is the main theorem of [43]. □

14 Interpolation

14.1 Linear Interpolation

The notion of interpolation between two Banach spaces is an old one, going back to work of Calderon and P. L. Lions. A systematic treatment of interpolation can be found in the treatise [50]. The basic idea of behind interpolation is simple. Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be two Banach spaces continuously embedded within a common topological vector space. Such a pair of spaces $\{\mathcal{X}_0, \mathcal{X}_1\}$ is said to be an *interpolation couple*. Given such an interpolation couple, one wishes to find a family of "intermediate" spaces $\mathcal{X}_\theta$, parametrized by $\theta \in (0, 1)$, so that if a linear operator $T : \mathcal{X}_0 + \mathcal{X}_1 \to \mathcal{X}_0 + \mathcal{X}_1$ restricts to a bounded operator $T : \mathcal{X}_i \to \mathcal{X}_i$, $i = 0, 1$, then T induces a bounded operator $T : \mathcal{X}_\theta \to \mathcal{X}_\theta$. There are many ways one may construct such a family of interpolation spaces, the two most common methods being the *complex interpolation* and the *real interpolation* methods.

While both are important, we will focus on the real interpolation method, as it is the only method of interpolation we need. There are several ways of defining this method, and we shall use the so called *K-method*. Given $\mathcal{X}_0$ and $\mathcal{X}_1$ two Banach spaces as above and $0 < t < \infty$, define

$$K(t, x) = \inf_{x=x_0+x_1} (\|x_0\|_{\mathcal{X}_0} + t\|x_1\|_{\mathcal{X}_1}), \quad x \in \mathcal{X}_0 + \mathcal{X}_1.$$

Observe that $\mathcal{X}_0 + \mathcal{X}_1$ is a Banach space under the norm $K(1, x)$, and that for every t, the norm $K(t, x)$ defines an equivalent norm on $\mathcal{X}_0 + \mathcal{X}_1$. Using this functional, then for any given $p \in [1, \infty]$, we can define a family of interpolation spaces $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$ as follows:

Definition 14.1 Let $\{\mathcal{X}_0, \mathcal{X}_1\}$ be an interpolation couple. Let $0 < \theta < 1$. If $1 \le p < \infty$, define

$$(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p} = \{x : x \in \mathcal{X}_0 + \mathcal{X}_1, \|a\|_{(\mathcal{X}_0,\mathcal{X}_1)_{\theta,p}} = \left(\int_0^\infty \left(t^{-\theta}K(t,a)\right)^p \frac{dt}{t}\right)^{1/p} < \infty\} \quad (14.1)$$

and if $p = \infty$, then define

$$(\mathcal{X}_0, \mathcal{X}_1)_{\theta,\infty} = \{x : x \in \mathcal{X}_0 + \mathcal{X}_1, \|a\|_{(\mathcal{X}_0,\mathcal{X}_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta}K(t,x) < \infty\}. \quad (14.2)$$

We collect some basic properties of these spaces.

Proposition 14.2 *Given an interpolation couple* $\{\mathcal{X}_0, \mathcal{X}_1\}$*,* $0 < \theta < 1$*, and* $1 \le p \le \infty$*, we have the following properties:*

(i) If $\mathcal{X}_0 = \mathcal{X}_1$*, then* $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p} = \mathcal{X}_0 = \mathcal{X}_1$.

(ii) If $\mathcal{X}_0 \subset \mathcal{X}_1$*, then for* $0 < \theta < \tilde{\theta} < 1$ *and* $1 \le p \le \tilde{p} \le \infty$*, we have*

$$(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p} \subset (\mathcal{X}_0, \mathcal{X}_1)_{\tilde{\theta},\tilde{p}}.$$

(iii) There exists a positive number $c_{\theta,p}$ *such that for all* $x \in \mathcal{X}_0 \cap \mathcal{X}_1$

$$\|x\|_{(\mathcal{X}_0,\mathcal{X}_1)_{\theta,p}} \le c_{\theta,p}\|x\|_{\mathcal{X}_0}^{1-\theta}\|x\|_{\mathcal{X}_1}^{\theta}.$$

(iv) Let $\{\mathcal{X}_0, \mathcal{X}_1\}$ *and* $\{\mathcal{Y}_0, \mathcal{Y}_1\}$ *be interpolation couples, and let* $T : \mathcal{X}_0 + \mathcal{X}_1 \to \mathcal{Y}_0 + \mathcal{Y}_1$ *be a linear map such that the* $T : \mathcal{X}_i \to \mathcal{Y}_i$ *are bounded,* $i = 0, 1$*. Then* $T : (\mathcal{X}_0, \mathcal{X}_1)_{\theta,p} \to (\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$ *is bounded and its operator norm is bounded by* $\|T\|_{\mathcal{X}_0\to\mathcal{Y}_0}^{1-\theta}\|T\|_{\mathcal{X}_1\to\mathcal{Y}_1}^{\theta}$.

The most classical examples of interpolation spaces are the usual Lebesgue spaces. Given $1 \le p_0, p_1 < \infty$, $0 < \theta < 1$, and p such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have that

$$(L^{p_0}(X), L^{p_1}(X))_{\theta,p} = L^p(X), \quad (14.3)$$

where X is any σ-finite measure space. In (14.3), one can replace the scalar valued Lebesgue spaces with vector-valued Lebesgue spaces, also known as Bochner spaces.

The literature on interpolation is vast and we will only develop as much of the tools as needed to study interpolation on the spaces of interest to us, namely, Besov spaces. To this end, we consider those pairs of interpolation spaces that have operators providing (essentially) the optimal decomposition of an element with respect to minimizing the K functional.

Definition 14.3 An interpolation couple $\{\mathcal{X}_0, \mathcal{X}_1\}$ is said to be *quasilinearizable* if there exists operators $V_j(t) \in L(\mathcal{X}_0 + \mathcal{X}_1, \mathcal{X}_j)$, $j \in \{0,1\}$, $t \in (0,\infty)$, such that

$$\begin{aligned}
V_0(t) + V_1(t) &= \mathrm{id}_{\mathcal{X}_0+\mathcal{X}_1} \\
\|V_0(t)x\|_{\mathcal{X}_0} &\leq c\|x\|_{\mathcal{X}_0}, && x \in \mathcal{X}_0 \\
\|V_1(t)x\|_{\mathcal{X}_1} &\leq ct^{-1}\|x\|_{\mathcal{X}_0}, && x \in \mathcal{X}_0 \\
\|V_0(t)x\|_{\mathcal{X}_0} &\leq ct\|x\|_{\mathcal{X}_1}, && x \in \mathcal{X}_1 \\
\|V_1(t)x\|_{\mathcal{X}_1} &\leq c\|x\|_{\mathcal{X}_1}, && x \in \mathcal{X}_1,
\end{aligned}$$

where the constant c is independent of x and t.

Lemma 14.4 *Let $\{\mathcal{X}_0, \mathcal{X}_1\}$ be a quasilinearizable interpolation couple. Then*

$$K(t,x) \leq \|V_0(t)x\|_{\mathcal{X}_0} + t\|V_1(t)x\| \leq 2cK(t,x). \tag{14.4}$$

Thus, we see that with the operators $V_0(t)$ and $V_1(t)$, we can write any $x \in \mathcal{X}_0 + \mathcal{X}_1$ as $x = V_0(t)x + V_1(t)x$, with such a decomposition yielding an optimal decomposition in computing the value of $K(t,x)$, up to some constant independent of t.

A convenient fact is that (vector-valued) Sobolev spaces provide quasilinearizable interpolation couples. Namely, if $\mathcal{X}$ is any Banach space, then $\{W^{k_0,p}(\mathbb{R}^n,\mathcal{X}), W^{k_1,p}(\mathbb{R}^n,\mathcal{X})\}$ is a quasilinearizable interpolation couple for all $1 \leq p \leq \infty$ and $0 \leq k_0 \leq k_1$.

Definition 14.5 Let $1 < p < \infty$ and $s > 0$. Define the Besov space

$$B^{s,p}(\mathbb{R}^n,\mathcal{X}) = (L^p(\mathbb{R}^n,\mathcal{X}), W^{k,p}(\mathbb{R}^n,\mathcal{X}))_{s/k,p}$$

where $k > s$ is any positive integer.

This, coincides with Definition 13.25 for Besov spaces. Furthermore, for any integer $m > s$, an equivalent norm is defined by

$$\|f\|_{B^{s,p}(\mathbb{R}^n,\mathcal{X})} = \|f\|_{L^p(\mathbb{R}^n,\mathcal{X})} + \left(\int_{\mathbb{R}^n}^{\infty} \left\||h|^{-s}\delta_h^m f\right\|_{L^p(\mathbb{R}^n,\mathcal{X})}^p \frac{1}{|h|^n} dh\right)^{1/p}, \tag{14.5}$$

just as in Proposition 13.4. For further reading on Besov spaces, see [2, 50].

By Proposition 14.2, an immediate consequence of a Besov space being an interpolation space is the following corollary:

Corollary 14.6 *Let $\mathcal{X}$ be a Banach space and $1 < p < \infty$. Let $T : L^p(\mathbb{R}^n,\mathcal{X}) \to L^p(\mathbb{R}^n,\mathcal{X})$ be a bounded linear operator that restricts to a bounded operator $T : W^{k,p}(\mathbb{R}^n,\mathcal{X}) \to W^{k,p}(\mathbb{R}^n,\mathcal{X})$. Then $T : B^{s,p}(\mathbb{R}^n,\mathcal{X}) \to B^{s,p}(\mathbb{R}^n,\mathcal{X})$ is bounded for any $0 < s < k$.*

This corollary is useful because it allows us to obtain estimates on Besov spaces from estimates on Sobolev spaces, the latter being much easier to obtain. While this corollary is tremendously useful, unfortunately it only applies to linear operators. Therefore, it is of interest to see how one might extend the above corollary to nonlinear operators.

14.2 Nonlinear Interpolation

There is a particular nonlinear interpolation result, due to Peetre [38], which will be useful for our purposes. Essentially, Peetre's result is that one can interpolate between Lipschitz operators.

Theorem 14.7 *[38, Theorem 2.1] Let $\{\mathcal{X}_0, \mathcal{X}_1\}$ and $\{\mathcal{Y}_0, \mathcal{Y}_1\}$ be two interpolation couples and let $T : \mathcal{X}_0 + \mathcal{X}_1 \to \mathcal{Y}_0 + \mathcal{Y}_1$ be any map. Let $\mathcal{X} = (\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$ and $\mathcal{Y} = (\mathcal{Y}_0, \mathcal{Y}_1)_{\theta,p}$ for some $0 < 1 < \theta$, $1 \le p \le \infty$. Next, let $\mathcal{D}_0$ and $\mathcal{D}_1$ be subsets of $\mathcal{X}_0 + \mathcal{X}_1$ on which we have the estimates*

$$\|Tx - Tx_0\|_{\mathcal{Y}_0} \le C\|x - x_0\|_{\mathcal{X}_0} \qquad \text{if } x - x_0 \in \mathcal{X}_0,\ x_0 \in \mathcal{D}_0 \tag{14.6}$$

$$\|Tx - Tx_1\|_{\mathcal{Y}_1} \le C\|x - x_1\|_{\mathcal{X}_1} \qquad \text{if } x - x_1 \in \mathcal{X}_1,\ x_1 \in \mathcal{D}_1, \tag{14.7}$$

where C is some constant. Then

$$\|Tx_0 - Tx_1\|_{\mathcal{Y}} \le C\|x_0 - x_1\|_{\mathcal{X}} \qquad \text{if } x_0 \in \mathcal{D}_0,\ x_1 \in \mathcal{D}_1,\ x_0 - x_1 \in \mathcal{X}. \tag{14.8}$$

In practice however, one rarely has a globally Lipschitz operator, i.e., an operator for which the estimates (14.6) and (14.7) hold for all x, x_0, x_1 satisfying the hypotheses. However, in certain instances, one can modify the proof of Theorem 14.7 to obtain a local interpolation theorem that satisfy the above estimates on a restricted domain. We present one such result below for Besov spaces, which is the only situation we will need. This result is needed to prove Lemma 10.8 in Part III.

Theorem 14.8 *Let $\mathcal{Z}$ be a Banach space. Let U_r be the open ball of radius r centered at the origin in $L^\infty(\mathbb{R}^n, \mathcal{Z})$. Let $\mathcal{X}_0 = W^{k_0,p}(\mathbb{R}^n, \mathcal{Z})$ and $\mathcal{X}_1 = W^{k_1,p}(\mathbb{R}^n, \mathcal{Z})$, where $k_1 > k_0$, and let $T : U_r \cap \mathcal{X}_0 \to \mathcal{X}_0$ be a map, $1 < p < \infty$. Suppose that T satisfies*

$$\|Tx - Ty\|_{\mathcal{X}_i} \le C\|x - y\|_{\mathcal{X}_i}, \qquad x, y \in U_r \cap \mathcal{X}_0, \quad x - y \in \mathcal{X}_i, \quad i = 0, 1,$$

where the Lipschitz constant C depends on $\|x\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}$, $\|y\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}$. Then there exists a constant $0 < \lambda < 1$, depending on $k_1 - k_0$, with the following significance. If $x, x_0 \in U_{\lambda r} \cap B^{s,p}(\mathbb{R}^n, \mathcal{X})$, where $k_0 < s < k_1$, then

$$\|Tx - Tx_0\|_{B^{s,p}(\mathbb{R}^n,\mathcal{Z})} \le \tilde{C}\|x - x_0\|_{B^{s,p}(\mathbb{R}^n,\mathcal{Z})}, \tag{14.9}$$

where $\tilde{C}$ is a constant depending on $k_1 - k_0$, $\|x\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}$, and $\|x_0\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}$.

Proof We know that $B^{s,p}(\mathbb{R}^n, \mathcal{Z}) = (\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$, where $s = (1-\theta)k_0 + \theta k_1$. Let $b = Tx - Tx_0$ and $a = x - x_0$. Thus, by the definition of the real interpolation functor, the theorem follows if we can show that $K(b, t) \le \tilde{C}K(a, t)$.

To this end, we use the fact that $(\mathcal{X}_0, \mathcal{X}_1)$ is a quasilinearizable interpolation couple. Hence, there exists operators $V_0(t)$ and $V_1(t)$ such that, setting $a_i(t) = V_i(t)a$, $i = 0, 1$, we have $a_i(t) \in \mathcal{X}_i$ and

$$\|a_0(t)\|_{\mathcal{X}_0} + \|a_1(t)\|_{\mathcal{X}_1} \le 2cK(a, t)$$

with c independent of t. In fact, the constant c only depends on $k_1 - k_0$. This is because, with our particular choice of $\mathcal{X}_0$ and $\mathcal{X}_1$, the $V_i(t)$ can be constructed explicitly (see [50, 1.13.2]). Moreover, this construction has the crucial property that

$$\|a_i(t)\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})} \le c\|a\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}.$$

So let $y(t) \in \mathcal{X}_0 + \mathcal{X}_1$ be such that

$$\begin{aligned} a_0(t) &= x - y(t) \\ a_1(t) &= y(t) - x_0. \end{aligned}$$

If we take $\lambda = \frac{1}{2(c+1)}$, then the hypotheses imply that $\|a\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}$ is sufficiently small so that $a_i(t), y(t) \in U_r$ for all t. Moreover, we have the estimate $\|y(t)\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})} \le c\|x - x_0\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})} + \|x\|_{L^\infty(\mathbb{R}^n,\mathcal{Z})}$. Then by definition of the K functional and using the hypotheses on T, we have

$$\begin{aligned} K(b,t) &\le \|Tx - Ty(t)\|_{\mathcal{X}_0} + t\|Ty(t) - Tx_0\|_{\mathcal{X}_1} \\ &\le C\|x - y(t)\|_{\mathcal{X}_0} + tC\|y(t) - x_0\|_{\mathcal{X}_1} \\ &= C(\|a_0(t)\|_{\mathcal{X}_0} + t\|a_1(t)\|_{\mathcal{X}_1}) \\ &\le 2CcK(a,t). \end{aligned}$$

Then $\tilde{C} = 2Cc$ is a constant depending on the appropriate quantities. This proves the theorem. □

15 Elliptic Boundary Value Problems

On a closed manifold, elliptic operators are automatically Fredholm when acting between suitable function spaces, say, Sobolev spaces. The way this is proved is by constructing a parametrix for the operator, which is an inverse for the operator modulo a compact operator. On a (compact) manifold with boundary, the construction of a parametrix requires extra data, namely a suitable choice of boundary conditions. The theory for constructing such boundary conditions is well understood and can be described naturally in terms of the pseudodifferential calculus of operators on the manifold and its boundary. A standard reference for this characterization is [19].

In this section, we first review the definition of a pseudodifferential operator and state the fundamental properties of the pseudodifferential algebra. Next, we define the notion of a (pseudodifferential) elliptic boundary condition and several theorems (Theorems 15.13, 15.19, and 15.25) concerning elliptic estimates for elliptic boundary value problems, in increasing order of strength. We present these theorems incrementally to illustrate the various degrees of sophistication needed for each of them. This is not just a mental exercise; in fact, our proof of Theorem 15.13, which involves the symbol calculus of pseudodifferential operators, lends itself to the analysis of the resolvent of an elliptic boundary value problem, which we pursue in Section 15.5. Theorem 15.19 is essentially due to [46], and its power draws from the wide range of parameters and function spaces for which it applies. Here, we obtain a stronger result than Theorem 15.13 because of the construction of certain special pseudodifferential operators, the Calderon projection and Poisson operator associated to an elliptic differential operator on a manifold with boundary. The Calderon projection is a projection onto the Cauchy data of the kernel of the associated elliptic operator. We also use this operator in Part I to study the tangent space to our monopole spaces. Because we will not need Theorem 15.19 in its full generality, we summarize the most relevant applications of Theorem 15.19 in the main body of this thesis as Corollary 15.22 for ease of reference. Finally, we have Theorem 15.25 for its application in obtaining elliptic bootstrapping on anisotropic function spaces in Part III.

The last part of this section, concerning the construction of a resolvent of an elliptic boundary value problem, is essentially a parameter-dependent version of the first half. Recall that the resolvent of an operator A acting between two Banach spaces is the operator $(A-\lambda)^{-1}$, where $\lambda \in \mathbb{C}$ is chosen such that the inverse exists. For an elliptic boundary problem, such a resolvent, when it exists, can be constructed out of λ-dependent pseudodifferential operators. We will thus need to introduce λ-dependent symbols and function spaces, after which we can state a particular resolvent estimate which we need for the analysis of Part III.

15.1 Pseudodifferential Operators

Let us recall the definition of a pseudodifferential operator (PSDO) on $\mathbb{R}^n$. For further reading, see e.g. [19], [59]. We begin our discussion by defining pseudodifferential operators in terms of double symbols, also known as amplitudes. That is, for every $m \in \mathbb{R}$, define the double symbol class $S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ to be the space of all smooth functions $a(x,y,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\sup_{x,y,\xi} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|} \tag{15.1}$$

for some constant $C_{\alpha,\beta}$ depending on the multi-indices α and β. In other words, $S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ is a Fréchet space topologized by the family of seminorms

$$\|a\|_{S^m_{\alpha,\beta}} := \langle \xi \rangle^{-m+|\alpha|} \sup_{x,y,\xi} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)|. \tag{15.2}$$

Given a double symbol $a(x,y,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, we obtain the associated mth order pseudodifferential operator $Op(a)$ given by

$$Op(a)f = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y) dy d\xi, \tag{15.3}$$

defined for $f \in C_0^\infty(\mathbb{R}^n)$. Let OS^m denote the class of all mth order pseudodifferential operators obtained by (15.3) for $a \in S^m$.

There is an equivalent characterization of pseudodifferential operators in terms of left and right symbols. That is, for every m consider the symbol class $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ consisting of the subspace of all functions $a(x,\xi) = a(x,y,\xi)$ in $S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ that are independent of $y \in \mathbb{R}^n$. From this, we can consider the associated left-quantized and right-quantized operators, respectively:

$$Op_l(a)f = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)f(y)dyd\xi, \tag{15.4}$$

$$Op_r(a)f = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(y,\xi)f(y)dyd\xi. \tag{15.5}$$

It turns out that the space of operators we obtain by left and right quantization of symbols $S^m(\mathbb{R}^n\times\mathbb{R}^n)$ is the same as the space of all mth order pseudodifferential operators obtained by quantizing double symbols in $S^m(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$. This follows from the following lemma:

Lemma 15.1 *Let $a\in S^m(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ be a double symbol of order m. Then there exist symbols $a_\ell, a_r\in S^m(\mathbb{R}^n\times\mathbb{R}^n)$ of order m such that*

$$Op(a) = Op_l(a_l) = Op_r(a_r).$$

The maps sending a to a_l and a_r define continuous maps from $S^m(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ to $S^m(\mathbb{R}^n\times\mathbb{R}^n)$, respectively.

Proof Our proof, which follows [59], proceeds as follows. First, we show that every amplitude can be expressed in terms of a left symbol. We have the following computation:

$$\begin{aligned}
Op(a)u &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi\\
&= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}\left((2\pi)^{-n}\int e^{i(y-z)\cdot\theta}a(x,z,\xi)dzd\theta\right)u(y)dyd\xi\\
&= (2\pi)^{-n}\int e^{i(x-y)\cdot(\xi-\theta)}\left((2\pi)^{-n}\int e^{i(x-z)\cdot\theta}a(x,z,\xi)dzd\theta\right)u(y)dyd\xi\\
&= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}\left((2\pi)^{-n}\int e^{i(x-z)\cdot\theta}a(x,z,\xi+\theta)dzd\theta\right)u(y)dyd\xi.
\end{aligned}$$

In passing from the second to third line above, we disregarded the order of integration and permuted exponential factors. This is justified in the sense of oscillatory integrals, i.e., such a formal manipulation is justified when the operator $Op(a)$ acts on a compactly supported smooth function u (see [59]). Altogether, the above computation shows us that if we define

$$a_\ell(x,\xi) = (2\pi)^{-n}\int e^{-i(x-z)\cdot(\xi-\theta)}a(x,z,\xi)dzd\theta,$$

then $Op_\ell(a_\ell) = Op(a)$. An analogous computation shows that

$$a_r(y,\xi) = (2\pi)^{-n}\int e^{-i(z-y)\cdot(\xi-\theta)}a(z,y,\xi)dzd\theta,$$

satisfies $Op_r(a_r) = Op(a)$.

It remains to show that, say, a_ℓ is a symbol of order m if A is a double symbol of order

m. For this, we write

$$a_\ell(x,\xi) = (2\pi)^{-n} \int e^{-iz\cdot\theta} a(x, x+z, \xi+\theta) dz d\theta,$$

we exploit the oscillatory nature of the exponential $e^{-iz\cdot\theta}$ along with the smoothness and decay properties of the symbol $a(x, x+z, \xi+\theta)$. More precisely, observe that we have the following identities:

$$\langle z \rangle^{-2N} \langle D_\theta \rangle^{2N} e^{-iz\cdot\theta} = \langle \theta \rangle^{-2M} \langle D_z \rangle^{2M} e^{-iz\cdot\theta} = e^{-iz\cdot\theta},$$

for all integers $N, M \geq 0$. Here, $\langle z \rangle^2 = 1 + |z|^2$, $\langle D_z \rangle^2 = 1 + (i\partial_z)^2$, and similarly for z replaced with θ. From these identities, a formal integration by parts implies that as an oscillatory integral, we have

$$a_\ell(x,\xi) = (2\pi)^{-n} \int e^{-iz\cdot\theta} \langle D_z \rangle^{2M} \langle \theta \rangle^{-2M} \langle D_\theta \rangle^{2N} \langle z \rangle^{-2N} a(x, x+z, \xi+\theta) dz d\theta. \qquad (15.6)$$

Using the fact that a is a symbol, then from the trivial inequalities

$$\langle \xi + \theta \rangle \leq \langle \xi \rangle \langle \theta \rangle$$
$$\langle \xi + \theta \rangle^{-1} \leq \langle \xi \rangle^{-1}$$

one easily deduces that the integrand of (15.6) is dominated by $C \langle z \rangle^{-2N} \langle \theta \rangle^{-2M+|m|} \langle \xi \rangle^m$, which is integrable for sufficiently large M and N. A similar bound applies to $\partial_x^\beta \partial_\xi^\alpha a_\ell$, where m is replaced with $m - |\alpha|$. It follows that $a_\ell(x,\xi) \in S^m$. Moreover, it readily follows from our computations that the map $a \mapsto a_\ell$ is continuous as a map from the space of double symbols to the space of symbols. □

Because of the above lemma, from now on, when we refer to the symbol of a pseudodifferential operator, we always mean its left symbol. We let $OS^m = OS^m(\mathbb{R}^n)$ denote the space of pseudodifferential operators of order m on $\mathbb{R}^n$.

We have the following standard theorem concerning pseudodifferential operators:

Theorem 15.2

(i) For all $m_1, m_2 \in \mathbb{R}$, we have the composition rule $OS^{m_1} \circ OS^{m_2} \to OS^{m_1+m_2}$.

(ii) If $P \in OS^0$, then P is bounded on $A^{s,p}(\mathbb{R}^n)$ for all $1 < p < \infty$ and $s \in \mathbb{R}$. Moreover, for any fixed s and p, the operator norm of P is bounded in terms of only finitely many symbol semi-norms $S^0_{\alpha,\beta}$.

Proof (i) Given two operators $A \in OS^{m_1}$ and $B \in OS^{m_2}$, we have that $A = Op_\ell(a)$ and $B = Op_r(b)$ for some $a \in S^{m_1}$ and $b \in S^{m_2}$ by the previous lemma. Thus, it follows straight from the definitions that $A \circ B = Op(a \cdot b)$, where $a \cdot b \in S^{m_1+m_2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. It follows that $A \circ B \in OS^{m_1+m_2}$.

(ii) This is a standard fact concerning pseudodifferential operators, see e.g. [48], [50]. □

Product-Type and Anisotropic Symbol Classes

We now define slightly more general symbol classes. These are the product-type and anisotropic symbol classes. These symbol classes reflect the types of operators that naturally arise in the context of anisotropic function spaces. Our goal is to prove composition and mapping properties for the operators associated to these symbol classes, analogous to those associated to the operators determined by standard symbols.

Suppose we have a decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. As before, we write $x, \xi \in \mathbb{R}^n$ as $(x^{(1)}, x^{(2)})$ and $(\xi^{(1)}, \xi^{(2)})$ with respect to this decomposition. Likewise, if $\alpha \in \mathbb{Z}_+^n$ is a multi-index of nonnegative integers, write $\alpha = (\alpha^{(1)}, \alpha^{(2)}) \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}$. For $m_1, m_2 \in \mathbb{R}$, we define the symbol class S^{m_1,m_2} to be the space of all smooth functions $a(x,\xi)$ such that

$$\sup_{x,\xi} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta} \left\langle \xi^{(1)} \right\rangle^{m_1 - |\alpha^{(1)}|} \left\langle \xi^{(2)} \right\rangle^{m_2 - |\alpha^{(2)}|}. \tag{15.7}$$

The space S^{m_1,m_2} is a Fréchet space whose topology is generated by the seminorms

$$\|a\|_{S^{m_1,m_2}_{\alpha,\beta}} := \sup_{x,\xi} \left\langle \xi^{(1)} \right\rangle^{-m_1 + |\alpha^{(1)}|} \left\langle \xi^{(2)} \right\rangle^{-m_2 + |\alpha^{(2)}|} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)|. \tag{15.8}$$

We define $OS^{m_1,m_2} = OS^{m_1,m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to be the class of all operators obtained via the formula (15.4) for $a \in S^{m_1,m_2}$. An operator in OS^{m_1,m_2} is called a *product-type pseudodifferential operator*.

For the purposes of generalizing Theorem 15.2 to our anisotropic spaces, we will need to introduce yet another type of symbol class. These symbols are "anisotropic symbols", since they obey an anisotropic type decay. Namely, given $m_1, m_2 \in \mathbb{R}$, define the symbol class $S^{(m_1,m_2)}$ to be the space of all smooth functions $a(x,\xi)$ such that

$$\sup_{x,\xi} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m_1 - |\alpha^{(1)}|} \left\langle \xi^{(2)} \right\rangle^{m_2 - |\alpha^{(2)}|}. \tag{15.9}$$

We define the seminorms $\| \cdot \|_{S^{(m_1,m_2)}_{\alpha,\beta}}$ on $S^{(m_1,m_2)}$ in the analogous way. Thus, when we differentiate symbols in $S^{(m_1,m_2)}$ in the $\xi^{(1)}$ variables, we get full radial decay in ξ, but we only get decay in $\xi^{(2)}$ when we differentiate in the $\xi^{(2)}$ derivatives. Hence, we have the containments, $S^0 \subset S^{(0,0)} \subset S^{0,0}$, where the symbol classes obey radial, anisotropic, and product type decay upon differentiation in the ξ variables, respectively. Define the class of anisotropic type operators $OS^{(m_1,m_2)} = OS^{(m_1,m_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ in the obvious way. In fact, all the operators in this paper will be of anisotropic type; we only consider them as product type, when applicable, in order to make use of the mapping properties of product type operators as established in [61]. For all $m \in \mathbb{R}$, we have the obvious inclusions

$$OS^m \subset OS^{(m,0)} \subset OS^{m,m}.$$

The proof of Lemma 15.1 follows through mutatis mutandis for our more general symbol class to prove the following:

Lemma 15.3 *Let $a_\ell \in S^{m_1,m_2}$. Then there exists a symbol $a_r \in S^{m_1,m_2}$ such that*

$$Op_\ell(a_\ell) = Op_r(a_r).$$

The map sending a_l and a_r defines a continuous map on S^{m_1,m_2}. The analogous results also hold for $S^{(m_1,m_2)}$.

In the above lemma, we omitted reference to product type and anisotropic type double symbols (defined in the obvious way) for simplicity, though the lemma also applies to such double symbols in the obvious way.

Theorem 15.4 *We have the following:*

(i) For all $m_1, m_1', m_2, m_2' \in \mathbb{R}$, we have the composition rules

$$OS^{m_1,m_2} \circ OS^{m_1',m_2'} \to OS^{m_1+m_1',m_2+m_2'}.$$
$$OS^{(m_1,m_2)} \circ OS^{(m_1',m_2')} \to OS^{(m_1+m_1',m_2+m_2')}.$$

(ii) If $P \in OS^{0,0}$ then P is a bounded operator on $A^{(s_1,s_2),p}$ for all $s_1, s_2 \in \mathbb{R}$ and $1 < p < \infty$. Moreover, for any fixed s_1,s_2,p, the operator norm of P is bounded in terms of only finitely many symbol semi-norms $S^{0,0}_{\alpha,\beta}$.

(iii) If $P \in OS^{(m_1,m_2)}$, then $P : A^{(s_1,s_2),p} \to A^{(s_1-m_1,s_2-m_2),p}$ is bounded. Moreover, the norm of P depends on only finitely many semi-norms $S^{(m_1,m_2)}_{\alpha,\beta}$.

Proof (i) This follows from the previous lemma as in the proof of Theorem 15.2 for the standard symbols. For (ii) and (iii), see [36], [59], [60]. □

Next, we recall the following standard fact. Given any sequence of symbols $a_{m_j} \in S^{m_j}$, with the m_j decreasing, $j \geq 0$, we can find a symbol $a \in S^m$ whose asymptotic expansion is given by the a_j. What this means is that for every $J \geq 0$, we have

$$a - \sum_{j=0}^{J} a_{m_j} \in S^{m_{J+1}}.$$

We also write

$$a \sim \sum_{j=0}^{\infty} a_{m_j}.$$

Define

$$OS^{-\infty} = \bigcap_{j \geq 0} OS^{-j}$$

to be the space of pseudodifferential operators with smooth integral kernel. Thus elements of $OS^{-\infty}$ are infinitely smoothing in that they map tempered distributions to C^∞ functions. Given two pseudodifferential operators S and T, we will write

$$S \equiv T$$

to denote $S - T \in OS^{-\infty}$.

Convention. Unless stated otherwise, all pseudodifferential operators in this paper will be *classic*, where a classic pseudodifferential operator of order m is an operator such that $a \sim \sum_{j=0}^{\infty} a_{m-j}$, with each a_{m-j} homogeneous in ξ of order $m-j$ on $|\xi| \geq 1$. Differential operators and their corresponding parametrices are all classic pseudodifferential operators. We will also allow pseudodifferential operators to be matrix-valued, since we ultimately want to consider pseudodifferential operators on vector bundles over manifolds. Note however that in this case, one needs to define the order of a pseudodifferential operator more carefully (see the next section).

In the next section, we will consider pseudodifferential operators on manifolds. By the use of a partition of unity, the notion of a pseudodifferential operator on $\mathbb{R}^n$ allows us to define pseudodifferential operators on compact manifolds in a natural way (see [59] for further background). We let $OS^m(X)$ denote the class of mth order pseudodifferential operators on X and similarly for the other operator classes.

15.2 The Basic Setup

Let X be a compact manifold with boundary ∂X and let E and F two vector bundles over X. Let $A : \Gamma(E) \to \Gamma(F)$ be an mth order elliptic differential operator mapping smooth sections of E to smooth sections of F. For all $s \in \mathbb{R}$ and $1 < p < \infty$, the operator A extends to a map

$$A : H^{s+m,p}(E) \to H^{s,p}(E) \tag{15.10}$$

where in general, the above map must be interpreted in the sense of distributions.

Remark 15.5 In our entire discussion of elliptic boundary value problems, the Bessel potential spaces $H^{s,p}$ on X can be replaced with the Besov spaces $B^{s,p}$. This is ultimately because pseudodifferential operators are bounded on both $H^{s,p}$ and $B^{s,p}$, and both these spaces have the same space of boundary values (again a Besov space) on ∂X. We write $H^{s,p}$ for specificity, and also because the letters A and B in $A^{s,p}$ and $B^{s,p}$ are already overworked in our notation.

When X is closed, the operator $A : H^{s+m,p}(E) \to H^{s,p}(F)$ is Fredholm. Let us quickly review why this is the case. The operator A is Fredholm because we can construct a *parametrix* for A, i.e. an approximate inverse $Q : H^{s,p}(F) \to H^{s+m,p}(E)$ that satisfies

$$AQ = \mathrm{id} + R_r, \tag{15.11}$$

$$QA = \mathrm{id} + R_l, \tag{15.12}$$

where $R_r, R_l \in OS^{-\infty}$ are smoothing errors. The operator Q is a classic pseudodifferential operator of order $-m$ constructed in local charts as follows.

Let $a := \sigma(A) = a_m + \ldots + a_0$ be the total symbol of A (we temporarily work in local chart modeled on an open subset of $\mathbb{R}^n$), with the a_j homogeneous of degree j. Since A is elliptic, we have $a_m(x, \xi)$ is invertible for $\xi \neq 0$. Hence, we can inductively solve the following system of equations on $|\xi| \geq 1$ for the symbols q_{-m-j}:

$$a_m q_{-m} = \text{id} \tag{15.13}$$

$$a_m q_{-m-j} + \sum_{\substack{\ell<j \\ k-|\alpha|-m-l=-j}} \frac{1}{\alpha!}\partial_\xi^\alpha a_k (i\partial_x)^\alpha q_{-m-l} = 0, \qquad j = 1, 2, \ldots \tag{15.14}$$

Then if we define a symbol q such that $q \sim \sum_{j=0}^{-\infty} q_{-m-j}$, then $\mathrm{Op}(q)$ gives us a desired local parametrix for A. Using a partition of unity argument, we may then construct an operator Q out of the local parametrices to obtain a global parametrix for A over the entire manifold X.

When ∂X is nonempty, the above construction only works in the interior of the manifold, and we have to consider boundary conditions for our operator. A boundary condition is a map B from the Cauchy data of sections of E to sections of another vector bundle, and given such a B, we may consider two operators: the full mapping pair (A, B) and the restricted operator A_B whose domain consists of those elements with Cauchy data annihilated by B. We wish to investigate those boundary conditions for which we may obtain the analogous Fredholm properties in the closed case. Moreover, we want our operators to yield elliptic estimates. The study of boundary conditions which fulfill this requirement goes back at least to the work of [1]. The approach we take will be that of [19] and [46].

Fix a collar neighborhood $[0, \epsilon) \times \partial X$ of X, where $t \in [0, \epsilon)$ is the inward normal coordinate and x denotes the coordinates on ∂X. In this neighborhood, write the principal part of A as $\sum_{j=0}^m A_j \partial_t^{m-j}$ where $A_j = A_j(x, t)$ are differential operators of degree j in the tangential variables. Let $(x, \xi) \in T^*\partial X \setminus \{0\}$. Consider the vector space of solutions $f : \mathbb{R}^+ \to \mathcal{C}$ to the ordinary differential equation

$$\left(\sum_{j=0}^m A_j(x, 0, \xi)\partial_t^{m-j}\right) f(t) = 0, \qquad t \in \mathbb{R}, \tag{15.15}$$

obtained by "freezing" A at $(x, 0, \xi)$. Here, $A_j(x, 0, \xi)$ is the symbol of A_j at $t = 0$. Let $M_x^{\pm}(\xi)$ denote the vector space of solutions to (15.15) which decay exponentially as $t \to \pm\infty$. The assumption that A is elliptic implies that the solution space of (15.15) decomposes as a direct sum $M_x^+(\xi) \oplus M_x^-(\xi)$, for all $(x, \xi') \in T^*\partial X \setminus 0$. Thus, letting E_x denote the fiber of E over $x \in \mathcal{X}$, we have an isomorphism $M_x^+(\xi) \oplus M_x^-(\xi) \cong E_x^m$ given by taking the full Cauchy data of a solution, $f(t) \mapsto (f(0), \ldots, \partial_t^{m-1} f(0))$. Via this isomorphism, we can identify $M_x^{\pm}(\xi) \subset E_x^m$.

Definition 15.6 For $(x, \xi) \in T^*\partial X \setminus 0$, define $\pi_A^+(x, \xi) : E_x^m \to E_x^m$ to be the projection onto $M_x^+(\xi)$ through $M_x^-(\xi)$.

From this projection, we can proceed to define what it means for a boundary condition B to be elliptic. Suppose we have $B = (B_1, \ldots, B_\ell)$ where

$$B_k : \Gamma(E_{\partial X})^m \to \Gamma(V_k)$$
$$B_k U := \sum_{j=0}^{m-1} b_{kj} U_j, \qquad U = (U_j)_{j=0}^{m-1},\ 1 \le k \le \ell,$$

where b_{kj} is a pseudodifferential operator mapping $\Gamma(E_{\partial X})$ to $\Gamma(V_k)$ for some vector bundle V_k over ∂X. Let $\beta_k = \max_j \{\operatorname{ord} b_{kj} + j\}$. Then the total boundary operator gives us a map

$$B : \Gamma(E_{\partial X})^m \to \oplus_{k=1}^{\ell} \Gamma(V_k) \tag{15.16}$$

Given a pseudodifferential operator T, let $\sigma_p(T)$ denote the principal symbol of T. If T is a matrix of pseudodifferential operators (where different entries of the matrix correspond to different vector bundles), then the order of T and hence its principal symbol need to be carefully defined. In the case of B above, since each B_k represents a boundary condition of order β_k, we define the principal symbol of B to be the following symbol-valued matrix:

$$\sigma_p(B) = (\sigma_{\beta_k - j} B_{kj})_{\substack{1 \le k \le \ell, \\ 0 \le j \le m-1}},$$

where $\sigma_i(B_{kj})$ is the usual principal symbol of B_{kj} if $B_{kj} \in Op^i(\partial X)$ and zero if $B_{kj} \in Op^{i'}(\partial X)$, $i' < i$.

Definition 15.7 Suppose the boundary operator B in (15.16) is such that $\sigma_p(B)(x,\xi) : (E_x)^m \to \oplus_{k=1}^{\ell} V_k$ restricted to $\operatorname{im} \pi_A^+(x,\xi)$ is an isomorphism onto $\operatorname{im} \sigma_p(B)(x,\xi)$ for all $(x,\xi) \in T^*\partial X \setminus 0$. Then B is an *elliptic boundary condition* for the operator A. In this case, we say that the pair (A, B) is elliptic.

The two most common elliptic boundary conditions that occur in practice are as follows. The first is when our elliptic boundary condition B is a local boundary condition, i.e., it is defined by a differential operator. Such a boundary condition B is also said to satisfy the Lopatinski-Shapiro condition.

Example 1. Let A be the Laplacian acting on scalar functions. We can take B to be

$$B_j : \Gamma(E_{\partial X}) \oplus \Gamma(E_{\partial X}) \to \Gamma(E_{\partial X}), \quad j = 0, 1$$

given by the first and second coordinate projections, respectively. These correspond to Dirichlet and Neumann boundary conditions, respectively. The boundary ODE (15.15) becomes the equation

$$\left(-\frac{d}{dt}^2 + |\xi|_x^2\right) f(t) = 0,$$

where $|\cdot|_x$ denotes the metric on $T_x^*\partial X$ induced from the Riemannian metric on X (we work in coordinates where the metric on $[0,\epsilon) \times \partial X$ is of the form $dt^2 + g_t^2$, g_t a Riemannian metric on ∂X). The spaces $M_x^{\pm}(\xi)$ are spanned by $e^{\mp|\xi|_x t}$, respectively, and passing to the

Cauchy data, we have $M_x^{\pm}(\xi) = \mathrm{span}\{(1, \mp|\xi|_x)\}$. Thus, we have

$$\pi_A^+(x,\xi) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2|\xi|_x} \\ \frac{|\xi|_x}{2} & \frac{1}{2} \end{pmatrix}.$$

It is now easy to see that the Lopatinsky-Shapiro condition holds for both the Dirichlet or Neumann boundary conditions. Hence, each of these boundary conditions is elliptic.

The second most common situation is when A is a Dirac operator and B is a zeroth order pseudodifferential operator.

Example 2. Let A be a Dirac operator and let B be the spectral projection onto the positive eigenspace of the tangential operator $A_0^{-1}A_1|_{t=0}$ associated to A. This is the Atiyah-Patodi-Singer (APS) boundary condition. Since the principal symbol of $A_0^{-1}A_1$ is self-adjoint, it follows that the positive and negative eigenspaces $M_x^{\pm}(\xi) \subset E_x$ are orthogonal. Thus, $\pi_A^+(x,\xi)$ is the orthogonal projection onto $M_x^{\pm}(\xi)$ and it is a basic fact that $\sigma_p(B) = \pi_A^+$. It immediately follows that B is an elliptic boundary condition.

Example 3. Let A be the div-grad-curl operator

$$A = \begin{pmatrix} *d & d \\ d^* & 0 \end{pmatrix}$$

acting on $\Gamma(E) = \Omega^1(Y) \oplus \Omega^0(Y)$ where Y is a 3-manifold with boundary Σ. In a collar neighborhood $[0,\epsilon) \times \Sigma$ of the boundary, we can write $a \in \Omega^1(Y)$ as $a = b + \alpha_1 dt$, where $b \in \Gamma([0,\epsilon);\Omega^1(\Sigma))$ and $\alpha_1 \in \Gamma([0,\epsilon);\Omega^0(\Sigma))$. Thus, restricting to the boundary, we have

$$\begin{aligned} \Gamma(E_\Sigma) &= \Omega^1(\Sigma) \oplus \Omega^0(\Sigma) \oplus \Omega^0(\Sigma) \\ (a,\alpha_0)|_\Sigma &= (b,\alpha_1,\alpha_0)|_\Sigma. \end{aligned}$$

The map A is a Dirac operator and its tangential operator $A_0^{-1}A_1|_{t=0}$ has the form

$$\begin{pmatrix} 0 & d_\Sigma & \check{*}d_\Sigma \\ d_\Sigma^* & 0 & 0 \\ -\check{*}d_\Sigma & 0 & 0 \end{pmatrix}$$

with respect to the above decomposition of $\Gamma(E_{\partial X})$, where d_Σ is the exterior derivative on Σ and $\check{*}$ is the Hodge star operator with respect to the induced metric on Σ.

Consider the following boundary operator

$$\begin{aligned} B = (B_1, B_0) : \Gamma(E_\Sigma) &\to \Gamma(\Omega^2(\Sigma)) \oplus \Gamma(\Omega^0(\Sigma)) \\ (b,\alpha_1,\alpha_0) &\mapsto (db, \alpha_1). \end{aligned}$$

A computation shows that B satisfies the Lopatinsky-Shapiro condition.

A similar computation shows that for $A = d + d^*$ acting on the total exterior algebra of differential forms on any manifold X with boundary, the tangential or normal component of the differential form at the boundary determines a local elliptic boundary condition.

Let us now see how an elliptic boundary condition allows us to construct a (left) parametrix for A. Let us assume X is an open subset of $\mathbb{R}^n$, with a smooth compact manifold ∂X as boundary. This case suffices, since by it is possible to transfer the result to a general compact manifold with boundary by using a partition of unity.

Let $u, v \in C^\infty(X)$. We have the following Green's formula[3]

$$\int_X (u, A^* v) = \int_X (Au, v) + \int_{\partial X} (Jru, rv), \tag{15.17}$$

where A^* is the formal adjoint of A, $r : \Gamma(E) \to \Gamma(E_{\partial X})^m$ is the map taking an element to its full Cauchy data of order $m-1$,

$$r(u) = (u(0), \ldots, \partial_t^{m-1} u(0)),$$

and $J : \Gamma(E)^m \to \Gamma(F)^m$ is the boundary endomorphism determined via integration by parts in (15.17). Thus, if we let u^0 denote the extension of u to $\mathbb{R}^n$ by zero, the above formula is equivalent to

$$Au^0 = (Au)^0 + r^* Jru, \tag{15.18}$$

in the sense of distributions. If we apply the (interior) parametrix Q to the above equation, we obtain

$$u^0 + R_\ell u^0 = Q(Au)^0 + Qr^* Jru, \tag{15.19}$$

where R_ℓ is the smooth error in (15.12). The last term in the above is the Green's potential, and it is the new term we need to control in the presence of a boundary. Taking the full trace of the Green's potential gives us the following operator:

Definition 15.8 Define $\tilde{P}^+ : \Gamma(E_{\partial X})^m \to \Gamma(E_{\partial X})^m$ by the formula

$$\tilde{P}^+ U = rQr^* JU, \qquad U = (u_0, \ldots, u_{m-1}).$$

The map $\tilde{P}^+$ is called an *approximate Calderon projection.*

Lemma 15.9 *The map $\tilde{P}^+$ is given by a matrix $(p_{ij}^+)_{i,j=0}^{m-1}$ of pseudodifferential operators p_{ij}^+ of order $i-j$. Its principal symbol $\sigma_p(\tilde{P}^+) = (\sigma_{i-j}(p_{ij}^+))$ is equal to π_A^+. Furthermore, $\tilde{P}^+$ is an approximate projection in the sense that $(\tilde{P}^+)^2 \equiv \tilde{P}^+$.*

Remark 15.10 In the next section, we will construct the Calderon projection P^+, which is a true projection with the same principal symbol as $\tilde{P}^+$.

Thus, an elliptic boundary condition B is one for which $\sigma_p(B) : \operatorname{im} \sigma_p(\tilde{P}^+) \to \operatorname{im} \sigma_p(B)$ is an isomorphism. Given an elliptic boundary condition B, we have the following fundamental lemma:

[3]In the main body of the text, we will usually drop the term r appearing in Green's formula, since for zeroth order Cauchy data, it is understood that the sections are being restricted to the boundary when performing the boundary integral. For now, since we deal with the possibly higher order Cauchy data, we include r in our notation.

Lemma 15.11 *(Parametrix on the boundary)*

(i) Suppose $\sigma_p(B)|_{\mathrm{im}\,\pi_A^+}$ is injective at all points $(x,\xi) \in T^*\partial X \setminus 0$. Then one can find matrices of pseudodifferential operators $Q_B' : \Gamma(E)^m \to \Gamma(E)^m$ and $Q_B : \oplus_{k=1}^{\ell}\Gamma(V_k) \to \Gamma(E)^m$ such that

$$Q_B' + Q_B B \equiv \mathrm{id}_{E^m} \tag{15.20}$$

$$Q_B B\tilde{P}^+ \equiv \tilde{P}^+ \tag{15.21}$$

$$Q_B'\tilde{P}^+ \equiv 0. \tag{15.22}$$

(ii) If $\sigma_p(B)|_{\mathrm{im}\,\pi_A^+}$ is bijective, then furthermore, $BQ_B \equiv \mathrm{id}_{\oplus_{k=1}^{\ell}\Gamma(V_k)}$ and the maps Q_B' and Q_B are uniquely determined modulo $OS^{-\infty}$.

This lemma follows from (slight modifications of) Proposition 20.1.5 and Theorem 19.5.3 in [19]. Let us unravel what this lemma says. The fact that B is an elliptic boundary condition means that $(1-\tilde{P}^+) \oplus B$ has injective symbol. Thus we can find operators Q_B'' and Q_B such that $Q_B'' \oplus Q_B$ is a left parametrix for $(1-\tilde{P}^+) \oplus B$. If we let $Q_B' = Q_B''(1-\tilde{P}^+)$, then Q_B' satisfies (15.22), and furthermore (15.20) is satisfied. Then (15.21) follows from (15.20) and (15.22). It follows that at the principal symbol level, $\sigma_p(Q_B)$ inverts the map $\sigma_p(B) : \mathrm{im}\,\sigma_p(\tilde{P}^+) \to \mathrm{im}\,\sigma_p(B)$, so that $\sigma_p(Q_B B) : E^m \to E^m$ is a projection onto $\mathrm{im}\,\sigma_p(\tilde{P}^+)$ through $\ker \sigma_p(B)$. Consequently, $\sigma_p(Q_B')$ is the complementary projection with range $\ker \sigma_p(B)$ and kernel $\mathrm{im}\,\sigma_p(\tilde{P}^+)$.

Let us see how the above analysis plays into the construction of a parametrix for the full mapping pair (A,B). On a manifold with boundary, the map $A : \Gamma(E) \to \Gamma(F)$ has an infinite dimensional kernel. In essence, the boundary condition B allows us to control the kernel, which means we can obtain an elliptic estimate for the full mapping pair (A,B). Moreover, if we consider the restricted operator A_B, then forcing elements to lie in the kernel of B means we have eliminated nearly all of the kernel of A. Consequently, the elliptic estimate for the full mapping pair (A,B) then gives us one for A_B.

We now work out the above considerations more precisely. As before, we return to the Euclidean setting $X \subset \mathbb{R}^n$ without loss of generality. Our main task is to control the term Qr^*Jru appearing on the right-hand-side of (15.19). Applying r to both sides, it follows from the definition of $\tilde{P}^+$ that

$$rQ(Au)^0 \equiv (1-\tilde{P}^+)ru.$$

From Lemma 15.11, we have $ru \equiv (Q_B'(1-\tilde{P}^+) + Q_B B)ru$, and thus, we have

$$\begin{aligned} u^0 &\equiv Q(Au)^0 + Qr^*J(Q_B'(1-\tilde{P}^+) + Q_B B)ru, \\ &\equiv (1 + Qr^*JQ_B'r)Q(Au)^0 + (Qr^*JQ_B)Bru. \end{aligned} \tag{15.23}$$

Observe that the above formula states that having knowledge of Au and Bru, we can recover u up to smoothing terms. Thus, the operators appearing in (15.23) provide us a left parametrix for the mapping pair (A, B), namely

$$\mathcal{Q}(f, g) = r_+[(1 + Qr^* J Q'_B r)Qf^0 + Qr^* J Q_B g], \tag{15.24}$$

where r_+ denotes the restriction of sections in $\mathbb{R}^n$ to X, i.e., it is the adjoint with respect to the extension by zero from X to $\mathbb{R}^n$.

Thus, to establish elliptic estimates for the full mapping pair (A, B), it suffices to establish suitable mapping properties of the parametrix $\mathcal{Q}$. Let $s \geq m$ and $1 < p < \infty$. We already know that $A : H^{s,p}(E) \to H^{s-m,p}(F)$. We have

$$r : H^{s,p}(E) \to \oplus_{j=0}^{m-1} B^{s-1/p-j,p}(E_{\partial X})$$

and from (15.16), we have

$$B : \oplus_{j=0}^{m-1} B^{s-1/p-j,p}(E_{\partial X}) \mapsto \oplus_{k=1}^{\ell} B^{s-1/p-\beta_k,p}(V_k) \tag{15.25}$$

For brevity, let

$$\begin{aligned} \mathcal{B}_m^{s-1/p,p} &= \oplus_{j=0}^{m-1} B^{s-1/p-j,p}(E_{\partial X}) \\ \mathcal{V}_{\bar\beta}^{s-1/p,p} &= \oplus_{k=1}^{\ell} B^{s-1/p-\beta_k,p}(V_k), \qquad \bar\beta = (\beta_1, \ldots, \beta_k). \end{aligned}$$

Theorem 15.12 *We have the following mapping properties:*

(i) (Transmission Property) Let $s \geq 0$. If $v \in H^{s,p}(E)$, then $r_+ Qv^0 \in H^{s+m,p}(E)$

(ii) (Approximate Poisson operator) Let $s \in \mathbb{R}$. If $U \in \mathcal{B}_m^{s-1/p,p}$, then $Qr^* JU \in H^{s,p}(E)$.

(iii) (Boundary parametrix) For every $s \in \mathbb{R}$, we have $Q_B : \mathcal{V}_{\bar\beta}^{s-1/p,p} \to \mathcal{B}_m^{s-1/p,p}$. Moreover, Q'_B is bounded on $\mathcal{B}_m^{s-1/p,p}$.

Part (iii) is automatic since it follows by construction from Lemma 15.11. The proofs of (i) and (ii) are involved and can be found in Hörmander, though more transparent expositions can be found in [59, Theorem 14.24] and [45, Lemma 4], respectively.

From this theorem, we now obtain one of our main theorems for elliptic boundary value problems (EBVP). We will state a slightly stronger version in the next section.

Theorem 15.13 *(EBVP - weak version) Let X be a compact manifold with boundary ∂X and let $A : \Gamma(E) \to \Gamma(F)$ be an mth order elliptic differential operator. Suppose B is an elliptic boundary condition satisfying (15.25). Let $1 < p < \infty$ and $s \geq 0$.*

(i) Let $u \in H^{t,p}(E)$, $t \geq 0$, and suppose $Au \in H^{s,p}(E)$ and $Bru \in \mathcal{V}_{\bar\beta}^{s-1/p,p}$. Then $u \in H^{s+m,p}(E)$ and

$$\|u\|_{H^{s+m,p}(E)} \leq C(\|Av\|_{H^{s,p}(E)} + \|Bru\|_{\mathcal{V}_{\bar\beta}^{s-1/p,p}} + \|u\|_{H^{t,p}(E)}). \tag{15.26}$$

(ii) The map $A_B : \{u \in H^{s+m,p}(E) : Bru = 0\} \to H^{s,p}(E)$ is Fredholm. Its kernel and cokernel are spanned by finitely many smooth sections.

(iii) If $\sigma_p(B) : E^m \to \oplus_{k=1}^{\ell} V_k$ is surjective, then the full mapping pair

$$(A, B) : H^{s+m,p}(E) \to H^{s,p}(E) \oplus \mathcal{V}_{\bar{\beta}}^{s-1/p,p}$$
$$u \mapsto (Au, Bru)$$

is a Fredholm operator.

Proof (i) This follows from the expression (15.24) for the parametrix $\mathcal{Q}$ for (A, B) and Theorem 15.12. (ii) From (i), we have an elliptic estimate for A_B since the boundary term Bru vanishes. This shows that A_B has closed range and finite dimensional kernel. The cokernel is finite dimensional because the adjoint problem is also an elliptic boundary value problem (see [46, Theorem 7]). (iii) Using Lemma 15.11(ii), one can show that the left parametrix we have constructed for (A, B) in (15.24) is also a right parametrix. Thus, (A, B), having a two-sided parametrix, is Fredholm. □

15.3 The Calderon Projection

There is a slightly cleaner approach to elliptic boundary value problems due to [46], in which we replace the approximate Calderon projection $\widetilde{P}^+$ of the previous section with a true projection P^+. Our goal in this section is to explain the relevant properties of the Calderon projection and some of its applications.

Let $A : \Gamma(E) \to \Gamma(F)$ be an mth order elliptic operator, which for simplicity, we take to be first order, though everything we discuss here generalizes straightforwardly for $m > 1$. Informally, the general picture is that following. We have two subspaces of interest, $\ker A$ and its restriction to the boundary $r(\ker A)$, where $r : \Gamma(E) \to \Gamma(E_\Sigma)$ is the restriction map. What we have is that there exists a pseudodifferential operator $P^+ : \Gamma(E_\Sigma) \to \Gamma(E_\Sigma)$ acting on boundary sections and a map $P : \Gamma(E_\Sigma) \to \Gamma(E)$ mapping boundary sections into the interior such that P^+ is a projection onto $r(\ker A)$ and the range of P is contained in $\ker A$. Furthermore, we have $rP = P^+$.

More precisely, and assigning the appropriate topologies to the spaces involved, let $s \in \mathbb{R}$ and $1 < p < \infty$, and let

$$Z^{s,p}(A) \subset H^{s,p}(E) \tag{15.27}$$

be the kernel of the operator $A : H^{s,p}(E) \to H^{s-1,p}(E)$. Let $Z_0(A)$ be the subset of $Z^{s,p}(A)$ consisting of those elements z with vanishing boundary values, i.e., $r(z) = 0$. Theorem 15.19 implies $Z_0(A) \subset C^\infty(E)$ and is finite dimensional. The map r extends to a bounded map $H^{s,p}(E) \to B^{s-1/p,p}(E_\Sigma)$ only when $s > 1/p$. However, if we restrict r to the kernel of A, it turns out that no such restriction on s is necessary. This is the content of the following very important theorem:

Theorem 15.14 *[45, 46] Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

(i) We have a bounded map $r : Z^{s,p}(A) \to B^{s-1/p,p}(E_{\partial X})$, and furthermore, its range is closed. In particular, if $Z_0(A) = 0$, then r is an isomorphism onto its image.

(ii) There is a pseudodifferential projection P^+ which projects $B^{s-1/p,p}(E_{\partial X})$ onto $r(Z^{s,p}(A))$. Furthermore, the principal symbol $\sigma_0(P^+)$ of P^+ is equal to the symbol π_A^+ (see Definition 15.6).

(iii) There is a map $P : B^{s-1/p,p}(E_{\partial X}) \to Z^{s,p}(A)$ whose range has $Z_0(A)$ as a complement. Furthermore, $PP^+ = P$ and $rP = P^+$.

Thus, in particular, the above theorem tells us that elements in the kernel of A of any regularity have well-defined restrictions to the boundary. In fact, the first part of Theorem 15.19(i) relies crucially on this fact.

Definition 15.15 The operators P^+ and P in Theorem 15.14 are called a *Calderon projection* and *Poisson operator* of A, respectively.

Remark 15.16 (i) From the definitions, it follows that P^+ is an elliptic boundary condition for A. (ii) A projection is defined not only by its range but also by its kernel. Thus, we have *a* Calderon projection and Poisson operator for A, since their kernels are not uniquely defined. When we speak of these operators then, we usually have a particular choice of these operators in mind. Seeley, for instance, has a particular construction of P^+ and P. However, it is usually only the range of P and P^+ that are of main interest to us, and these are uniquely specified by the above definitions. Hence, a Calderon projection is often times referred to as *the* Calderon projection in the literature.

Altogether, P^+ is a projection onto the Cauchy data of the kernel of A, and P is a map from the Cauchy data of the kernel into the kernel. The latter map is an isomorphism when $Z_0(A) = 0$. Furthermore, we have

Corollary 15.17 *For all $s \in \mathbb{R}$, smooth configurations are dense in $Z^{s,p}(E)$. Furthermore, suppose $s > 1/p$. Then $Z^{s,p}(E) \subset H^{s,p}(E)$ is complemented. Moreover, if $Z_0(A) = 0$, then $Pr : H^{s,p}(E) \to Z^{s,p}(A)$ is a bounded projection onto $Z^{s,p}(A)$.*

Proof We have that $Z^{s,p}(A)$ is the direct sum of $Z_0(A)$ and the image of $P : B^{s-1/p,p}(E_{\partial X}) \to H^{s,p}(E)$. The first statement now follows since the space $Z_0(A)$ is spanned by smooth sections and smooth sections are dense in $B^{s-1/p,p}(E_{\partial X})$. Now consider $s > 1/p$. Then the map $Pr : H^{s,p}(E) \to Z^{s,p}(A)$ is a projection onto the image of P, which is of finite codimension in $Z^{s,p}(A)$. From this, one can construct a projection of $H^{s,p}(E)$ onto $Z^{s,p}(A)$, which means $Z^{s,p}(A)$ is a complemented subspace. If $Z_0(A) = 0$, then the range of P is all of $Z^{s,p}(A)$, whence Pr is a projection onto $Z^{s,p}(A)$. □

The operators P^+ and P play a fundamental role in the study of elliptic boundary value problems. We will use them to prove a stronger version of Theorem 15.13 in this section. But before doing so, we first present an important application of these operators.

Let A be a first order formally self-adjoint elliptic operator. Then the operator $J := A_0$ in (15.15) is a skew-symmetric automorphism on the boundary, and Green's formula (3.66) for A defines for us a symplectic form

$$\omega(u, v) = \int_\Sigma (u, -Jv)$$

on boundary sections $u, v \in \Gamma(E_{\partial X})$. This symplectic form extends to a well-defined symplectic form on $B^{s,p}(E_{\partial X})$ for $(s, p) = (0, 2)$ and for $s > 0$, $p \geq 2$, and the map $-J$ is a compatible complex structure with respect to this symplectic form. Indeed, for this range of s and p, we have $B^{s,p}(E_{\partial X}) \hookrightarrow L^2(E_{\partial X})$, with the latter a strongly symplectic Hilbert space.

We say that a closed subspace of $B^{s,p}(E_{\partial X})$ is Lagrangian if it is isotropic with respect to ω and it has an isotropic complement. Observe that if $L \subset L^2(E_{\partial X})$ is Lagrangian, then JL is a Lagrangian complement of L.

Proposition 15.18 *[4] Let A be a Dirac operator. Then* $\operatorname{im} P^+$ *and* $J \operatorname{im} P^+$ *are complementary Lagrangian subspaces of* $B^{s,p}(E_{\partial X})$*, where* $(s,p) = (0,2)$ *or* $s > 0$*,* $p \geq 2$*.*

Proof In [4], it is shown that $\operatorname{im} P^+$ and $J \operatorname{im} P^+$ define complementary Lagrangian subspaces of $L^2(E_{\partial X})$. Here, it is essential that one uses the trick of constructing an "invertible double" for the operator A. However, since P^+ is a pseudodifferential projection, it is bounded on $B^{s,p}(E_{\partial X})$. Without loss of generality, we can suppose P^+ is an orthogonal projection (making a projection into an orthogonal projection preserves the property of being pseudodifferential). Define $P^- = JP^+J^{-1}$. Then $\operatorname{im} P^- = J \operatorname{im} P^+$ and its principal symbol agrees with the principal symbol of $1 - P^+$. It follows that $\operatorname{im} P^+ \oplus J \operatorname{im} P^+$ is a closed subspace of $B^{s,p}(E_{\partial X})$ of finite codimension. We now apply Lemma 18.5, which tells us that $\operatorname{im} P^+ \oplus J \operatorname{im} P^+$ is in fact all of $B^{s,p}(E_{\partial X})$. $\square$

We now apply our Calderon projection and Poisson operator to the study of general elliptic boundary value problems. Using these operators, Seeley in [46] constructs a parametrix similar to (15.24) for elliptic operators with elliptic boundary conditions. From this, he even achieves stronger results than those of Theorem 15.19. This is because Theorem 15.14 holds for all $s \in \mathbb{R}$ and so we may extend Theorem 15.19 to certain negative parameters. For $\kappa = 0, 1, \ldots, m-1$, let

$$r_\kappa : H^{s,p}(E) \to \oplus_{j=0}^{\kappa} B^{s-1/p-j,p}(E_k)$$

be the trace map onto the Cauchy data up to order κ.

Theorem 15.19 *(EBVP - strong version, [46, Theorem 4]) Let X be a compact manifold with boundary ∂X and let $A : \Gamma(E) \to \Gamma(F)$ be an mth order elliptic differential operator. Suppose B is an elliptic boundary condition satisfying (15.25) which depends only on the Cauchy data up to order κ. Let $1 < p < \infty$ and $s > -m + \kappa + 1/p$.*

(i) Let $u \in H^{t,p}(E)$, $t \in \mathbb{R}$, and suppose $Au \in H^{s,p}(E)$ and $Br_\kappa u \in \mathcal{V}_{\bar{\beta}}^{s-1/p,p}$. Then $u \in H^{s+m,p}(E)$ and

$$\|u\|_{H^{s+m,p}(E)} \leq C(\|Av\|_{H^{s,p}(E)} + \|Bru\|_{\mathcal{V}_{\bar{\beta}}^{s-1/p,p}} + \|u\|_{H^{t,p}(E)}). \tag{15.28}$$

(ii) The map $A_B : \{u \in H^{s+m,p}(E) : Bru = 0\} \to H^{s,p}(E)$ is Fredholm. Its kernel and cokernel are spanned by finitely many smooth sections.

(iii) If $\sigma_p(B) : E^m \to \oplus_{k=1}^{\ell} V_k$ is surjective, then the full mapping pair

$$(A,B) : H^{s+m,p}(E) \to H^{s,p}(E) \oplus \mathcal{V}_{\bar{\beta}}^{s-1/p,p}$$
$$u \mapsto (Au, Br_\kappa u)$$

is a Fredholm operator.

Remark 15.20 Seeley only proves this theorem for $p = 2$, but because all the maps involved are pseudodifferential or involve taking traces or extensions, the generalization to $1 < p < \infty$ is automatic. See also Remark 15.5.

Remark 15.21 By a standard argument, the lower order term $\|u\|_{H^{t,p}(E)}$ in (15.28) can be replaced with $\|\pi u\|$, where π is any projection onto the finite dimensional space of solutions to $Au = Br_\kappa u = 0$, and $\|\cdot\|$ is any norm on that space. In other words, we need only control the kernel of the operator (A, B) to get the estimate (15.28). In particular, if (A, B) has no kernel, the term $\|u\|_{H^{t,p}(E)}$ can be omitted.

We will not need Theorem 15.19 in its full generality. For convenience, we summarize the particular applications we have in mind in the below corollary. Furthermore, as mentioned in Remark 15.5, all instances of the $H^{s,p}$ topology occurring in Theorems 15.13, 15.19, 15.14 can be replaced by $B^{s,p}$. We thus have the following results for Besov spaces, which we will primarily use in Part I when $X = Y$ is a 3-manifold.

Corollary 15.22 *We have the following elliptic boundary value problems:*

(i) Let $A = \Delta$ be the Laplacian acting on scalar functions. Then the Dirichlet and Neumann boundary conditions are elliptic boundary conditions. For the Dirichlet problem, we have the elliptic estimate

$$\|u\|_{B^{s+2,p}(X)} \le C(\|\Delta u\|_{B^{s,p}(X)} + \|r_0 u\|_{B^{s+2-1/p,p}(\partial X)}) \tag{15.29}$$

for $s + 2 > 1/p$. For the Neumann problem, we have the elliptic estimate

$$\|u\|_{B^{s+2,p}(X)} \le C\left(\|\Delta u\|_{B^{s,p}(X)} + \|r_0(\partial_\nu u)\|_{B^{s+1-1/p,p}(X)} + \left|\int_X u\right|\right) \tag{15.30}$$

for $s + 2 > 1 + 1/p$, where $\partial_\nu u$ denotes the derivative of u with respect to the outward unit normal to ∂X.

(ii) Let $A = d + d^*$ be the Hodge operator acting on $\oplus_{i=0}^n \Omega^i(X)$, the exterior algebra of differential forms on X. Then the tangential component[4] $a|_{\partial X}$ and normal component $*a|_{\partial X}$ are elliptic boundary conditions. In particular, if $a \in \Omega^1(X)$, then we have the elliptic estimate

$$\|a\|_{B^{s+1,p}\Omega^1(X)} \le C(\|da\|_{B^{s,p}\Omega^2(X)} + \|d^*a\|_{B^{s,p}\Omega^0(X)} + \|a^h\|_{B^{s,p}\Omega^1(X)}) \tag{15.31}$$

for $s + 1 > 1/p$, where a^h denotes the orthogonal projection of a onto the space

$$H^1(X, \partial X; \mathbb{R}) \cong \{a \in \Omega^1(X) : da = d^*a = 0, a|_{\partial X} = 0\}. \tag{15.32}$$

(iii) Let $A : \Gamma(E) \to \Gamma(F)$ be a Dirac operator. If B is any pseudodifferential projection onto $r(\ker A)$, then B is an elliptic boundary condition for A. We have the elliptic

[4]See footnote 4 in Part I.

estimate

$$\|u\|_{B^{s+1,p}(E)} \leq C(\|Au\|_{B^{s,p}(F)} + \|Bru\|_{B^{s+1-/1p,p}(E_{\partial X})}). \tag{15.33}$$

for $s+1 > 1/p$.

Proof For (i), a standard computation shows that the kernel of the Dirichlet Laplacian is zero, and the kernel of the Neumann Laplacian is spanned by constant functions. We now apply Remark 15.21. For (ii), the kernel of $d+d^*$ on $\Omega^1(Y)$ with vanishing tangential component is the space (15.32). We now apply the previous theorem and Remark 15.21. Observe that for the Dirichlet Laplacian we took $\kappa = 0$ in Theorem 15.19. For (iii), there is no term to account for the kernel due to unique continuation, Theorem 17.1, which implies that r maps $\ker A$ isomorphically onto its image (hence Br maps $\ker A$ isomorphically onto its image). □

15.4 Generalizations

We wish to generalize Theorem 15.19 even further. We have two specific reasons for this. First, we wish to obtain elliptic estimates on anisotropic function spaces. Second, we want to consider boundary conditions that are not necessarily pseudodifferential. It turns out that these considerations do not pose a significant obstacle to generalizing the previous results. In short, this is because pseudodifferential operators are bounded on anisotropic function spaces, and moreover, because the key properties that make a boundary condition elliptic is essentially a functional analytic property, not a pseudodifferential property.

For simplicity, let us take A to be first order (the only case we will need in this paper), though what follows easily generalizes to elliptic operators of any order. Observe then that a boundary condition for A is simply a choice subspace of $\mathcal{U} \subset \Gamma(E_\Sigma)$ of the boundary data space, where $E_\Sigma = E|_\Sigma$. The desirable boundary conditions are those for which the operator

$$A_{\mathcal{U}} : \{x \in \Gamma(E) : r(x) \in \mathcal{U}\} \to \Gamma(F) \tag{15.34}$$

is a Fredholm operator in the appropriate function space topologies. In typical situations, like the ones considered previously, the subspaces $\mathcal{U}$ are given by the range of pseudodifferential projections. However, from the above viewpoint, one need only consider the functional analytic setup of subspaces and appropriate function space topologies in order to understand the operator (15.34).

Let X be a manifold with boundary, and suppose it can be written as a product $X = X_1 \times X_2$, where X_1 is a compact manifold with boundary and X_2 is closed. We have the anisotropic Besov spaces $B^{(s_1,s_2),p}(X_1 \times X_2)$ on X and $B^{(s_1,s_2),p}(\partial X_1 \times X_2)$ on ∂X, for $s_1 \geq 1$, $s_2 \geq 0$, and $1 < p < \infty$, as defined in Definition 13.21. These spaces induce topologies on vector bundles, and so in particular, we have the spaces $B^{(s_1,s_2),p}(E)$, $B^{(s_1,s_2),p}(F)$ and such. We have a restriction map

$$r : B^{(s_1,s_2),p}(E) \to B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2}),$$

and given a subspace

$$\mathcal{U} \subset B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2}),$$

we get the space

$$B_{\mathcal{U}}^{(s_1,s_2),p}(E) = \{x \in B^{(s_1,s_2),p}(E) : r(x) \in \mathcal{U}\}.$$

Thus, (15.34) yields the operator

$$A_{\mathcal{U}} : B_{\mathcal{U}}^{(s_1,s_2),p}(E) \to B^{(s_1-1,s_2),p}(F). \tag{15.35}$$

We can now ask how properties of $\mathcal{U}$ correspond with properties of the induced map (15.35).

For this, we must distinguish a special subspace of $B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2})$, namely $\operatorname{im} P^+$, where

$$P^+ : B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2}) \to B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2})$$

is the Calderon projection of A. This is a projection onto $r(\ker A)$, the boundary values of the kernel of A. As noted, P^+ is a pseudodifferental projection. By Theorem 15.2, we know that pseudodifferential operators are bounded on anisotropic Besov spaces. Thus, $\operatorname{im} P^+$ is a well-defined closed subspace of the boundary data space $B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2})$. The relevant properties for the operator (15.35) can now be understood via the relationship between $\mathcal{U}$ and $\operatorname{im} P^+$.

Recall that we have the notion of two subspaces of a Banach space being Fredholm (see Section 18). We have the following theorem:

Theorem 15.23 *[36] Let*[5] $s_1 \geq 1$, $s_2 \geq 0$, *and* $1 < p < \infty$ *and consider the operator* $\mathcal{A}_{\mathcal{U}}$ *given by (15.35).*

(i) The operator $A_{\mathcal{U}}$ *is Fredholm if and only if* $\mathcal{U}$ *and* $\operatorname{im} P^+$ *are Fredholm.*

(ii) The kernel of $A_{\mathcal{U}}$ *is spanned by finitely many smooth configurations if and only if* $\mathcal{U} \cap \operatorname{im} P^+$ *is spanned by finitely many smooth configurations.*

(iii) The range of $A_{\mathcal{U}}$ *is complemented by the span of finitely many smooth configurations if and only if* $U + \operatorname{im} P^+$ *is complemented by the span of finitely many smooth configurations.*

Given a subspace $\mathcal{U} \subset B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2})$ Fredholm with $\operatorname{im} P^+$, we can thus construct a projection

$$\Pi_{\mathcal{U}} : B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2}) \to B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2}) \tag{15.36}$$

with $\operatorname{im} \Pi_{\mathcal{U}} = \mathcal{U}$ and $\ker \Pi_{\mathcal{U}}$ equal to $\operatorname{im} P^+ = r(\ker A)$ up to some finite dimensional space.

We now have the following notion of a general elliptic boundary condition:

Definition 15.24 A boundary condition B for the first order elliptic operator A with domain $B^{(s_1,s_2),p}(X_1 \times X_2)$ is a map

$$B : B^{(s_1-1/p,s_2),p}(E_{\partial X_1 \times X_2}) \to \mathcal{X} \tag{15.37}$$

from the boundary data space to some Banach space $\mathcal{X}$. We say that B is *elliptic* if $B : \operatorname{im} P^+ \to \mathcal{X}$ is Fredholm.

[5] To keep matters simple, we state the hypotheses for $s_1 \geq 1$, which is all we need in this paper. This is in contrast to [34], where we needed to consider spaces of lower regularity than the order of the operator.

Given $\mathcal{U}$ and $\Pi_{\mathcal{U}}$ as above, the complementary projection $1 - \Pi_{\mathcal{U}}$ is then an elliptic boundary condition in the above sense, since $(1 - \Pi_{\mathcal{U}}) : \operatorname{im} P^+ \to \operatorname{im}(1 - \Pi_{\mathcal{U}})$ is Fredholm. Using this, we can obtain an elliptic estimate for the full mapping pair $(A_{\mathcal{U}}, 1 - \Pi_{\mathcal{U}})$.

Theorem 15.25 *[36] Let $s_1 \geq 1$, $s_2 \geq 0$, $1 < p < \infty$, and suppose $\mathcal{U} \subseteq B^{(s_1-1/p,s_2)}(E_{\partial X_1 \times X_2})$ is Fredholm with $r(\ker A)$. Then consider the full mapping pair*

$$(A_{\mathcal{U}}, (1 - \Pi_{\mathcal{U}})r) : B^{(s_1,s_2),p}(E) \to B^{(s_1-1,s_2),p}(F) \oplus B^{(s_1-1/p,s_2),p}(E_{\partial X_1 \times X_2}). \tag{15.38}$$

This operator is Fredholm and we have the elliptic estimate

$$\|u\|_{B^{(s_1,s_2),p}(X_1\times X_2)} \leq C\left(\|Au\|_{B^{(s_1-1,s_2),p}(X_1\times X_2)} + \|(1-\Pi_{\mathcal{U}})ru\|_{B^{(s_1-1/p,s_2),p}(\partial X_1\times X_2)} + \|\pi u\|\right) \tag{15.39}$$

Here π is any projection onto the finite dimensional kernel of (15.38) and $\|\cdot\|$ is any norm on that space.

When $s_2 = 0$, (15.39) is the usual elliptic estimates on isotropic spaces. Thus, the significance of the above theorem is that tangential anisotropy is preserved. We will need the above theorems in Part III when we consider anisotropic spaces on the cylindrical 4-manifold $\mathbb{R} \times Y$ and boundary conditions are supplied by (nearly pseudodifferential) projections onto certain Banach subspaces of the boundary configuration space.

15.5 The Resolvent of an Elliptic Boundary Value Problem

In the previous section, we proved that boundary conditions satisfying certain properties allow us to obtain elliptic estimates for elliptic operators on a variety of function spaces. In this section, we use the tools developed thus far to consider the resolvent of an elliptic boundary value problem. In short, what this amounts to is that we must consider a parameter dependent version of the operators constructed in the previous section, the parameter being the resolvent parameter under consideration. Our goal is to construct the resolvent of an elliptic boundary value problem using the parameter dependent pseudodifferential calculus. Furthermore, we want to estimate the operator norm of this resolvent, on suitable function spaces, as the resolvent parameter tends to infinity. Thus, the main technical point is to make sure that the construction of all operators involved depend on λ in a controlled or uniform way.

There is a well developed theory for studying parameter dependent pseudodifferential operators and their applications to elliptic boundary value problems due to Grubb and Seeley (see [47], [18], [16], [17]).[6] Using the tools contained in their works, our goal is to obtain a particular resolvent estimate in Theorem 15.32 and Corollary 15.34. This resolvent estimate will be used in Part III in order to obtain elliptic estimates for operators of the form $\frac{d}{dt} + D$ on the cylinder $\mathbb{R} \times Y$ with D a formally self-adjoint operator on Y. Here, we apply the Fourier transform in the $\mathbb{R}$ variable and thus obtain a resolvent, and it is a resolvent of an elliptic boundary value problem, since the operator D is supplied with boundary conditions.

[6] In fact, Grubb and Seeley define a quite general parameter dependent symbol class in order to study the asymptotic expansions of traces of parameter dependent operators. In the language of [18], we only need to consider strongly-polyhomogeneous symbol class rather than more the general symbol class.

To minimize notation and because it is the only situation that interests us, in what follows, we restrict to first order operators A, though what we do can certainly be generalized to higher order operators.

Parameter Dependent PSDOs and Sobolev Spaces

We define parameter-dependent symbol classes, where λ is a complex parameter with values in some open subset $\Gamma \subset \mathbb{C}$. We consider the case when Γ is an open sector in $\mathbb{C} \setminus \{0\}$, i.e., it is an open set closed under positive scaling. The type of λ-dependent symbols on $\mathbb{R}^n$ we consider are those which are homogeneous in both the cotangent variables and λ. More precisely, we consider the symbol class $S^m_{hg}(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$ consisting of those functions $a(x, \xi, \lambda)$ such that

(i) letting $\zeta = (\xi, \lambda)$, the estimate

$$\sup_{x,\zeta} |\partial_x^\beta \partial_\zeta^\alpha a(x, \xi, \lambda)| \leq C_{\alpha\beta} \left\langle (\xi, \lambda) \right\rangle^{m-|\alpha|}$$

holds for all multi-indices α and β;

(ii) we have

$$a(x, \sigma\xi, \sigma\lambda) = \sigma^m a(x, \xi, \lambda)$$

for all $\sigma > 0$ and $|(\xi, \lambda)| \geq 1$.

Thus, (i) says that a is essentially a symbol in $n+1$ Fourier variables, where we regard the last Fourier variable as a complex parameter instead.[7] Condition (ii) means that we require homogeneity of a outside the unit sphere in $\mathbb{R}^n \times \Gamma$. We say that a symbol a is *strongly polyhomogeneous* of degree m if $a \sim \sum_{j=0}^\infty a_{m-j}$, where $a_{m-j} \in S^{m-j}_{hg}(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$. That is, for every $N \geq 0$, we have

$$\sup_{x,\zeta} \left| \partial_x^\beta \partial_\zeta^\alpha \left(a - \sum_{j=0}^N a_{m-j} \right) \right| = O\Big(\left\langle (\xi, \lambda) \right\rangle^{m-N-1-|\alpha|} \Big), \tag{15.40}$$

for all multi-indices α,β. We denote the class of such symbols by $S^m_{sphg}(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$. We write $OS^m_{sphg} = OS^m_{sphg}(\mathbb{R}^n)$ to denote the space of all pseudodifferential operators obtained from the symbol space $S^m_{sphg}(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$.

The model example of an element of $S^m_{hg} = S^m_{hg}(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$ is the symbol $a(x, \xi, \lambda) = \langle (\xi, \lambda) \rangle^m$. Another example, and the one most relevant for us, is constructed as follows. Let $a(x, \xi)$ be the principal symbol of a self-adjoint first order elliptic differential operator. Let $\Gamma = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z) \pm \pi/2| < \epsilon\}$ be a small sector centered along the imaginary axis. Then $a(x, \xi, \lambda) := (a(x, \xi) - i\lambda)^{-1}$ for $\lambda \in \Gamma$ and $|(\xi, \lambda)| \geq 1$ defines for us an element of S^{-1}_{hg}.

Finally, we say that a symbol is *weakly homogeneous* of degree k if it is positively homogeneous of degree k only on the cylinder $|\xi| \geq 1$ inside $\mathbb{R}^n \times \Gamma$. The notion of weakly polyhomogeneous follows accordingly.

[7] In everything that we do, we do not require any differentiability of our symbols in λ. For many other applications however, such as those considered by Grubb and Seeley, one does need differentiability in λ for the λ-dependent symbol calculus.

Remark 15.26 For our purposes, it is actually not necessary to consider those parameter dependent PSDO's which are strongly polyhomogeneous. One of the goals of [18] is to obtain asymptotic formulas for the trace of parameter dependent PSDO's, in which case, polyhomogeneity of the operator becomes relevant. With the purposes we have in mind, all that is necessary for our PSDO's is that (15.40) holds for $N = 0$ and all multi-indices α,β. Nevertheless, most (but not all) of the parameter dependent PSDO's we construct will be strongly polyhomogeneous since they will be (symbolically) constructed out of powers of the operator $A - i\lambda$.

Naturally, the function spaces on which the λ-dependent pseudodifferential are bounded (uniformly in λ) are those which are themselves λ-dependent. Thus, define[8]

$$H^s_\lambda(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^s_\lambda} = \| \langle (\xi, \lambda) \rangle^s \hat{f}(\xi) \|_{L^2} < \infty \right\}, \qquad s \in \mathbb{R}.$$

Given a map $T : H^s(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$, possibly depending on λ, we write

$$T : H^s_\lambda(\mathbb{R}^n) \to H^t_\lambda(\mathbb{R}^n)$$

to denote that the operator T is bounded uniformly in λ, for all $\lambda \in \Gamma$.

We have the following theorem, which is proved in exactly the same manner as Theorem 15.2.

Theorem 15.27 *(i) We have the composition rule*

$$OS^{m_1}_{sphg} \circ OS^{m_2}_{sphg} \to OS^{m_1+m_2}_{sphg}.$$

(ii) If $T \in OS^m_{sphg}(\mathbb{R}^n)$ then $T : H^s_\lambda(\mathbb{R}^n) \to H^{s-m}_\lambda(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

We also have a λ-dependent generalization of the basic trace and extension theorems.

Lemma 15.28 *(i) Let $s > m + 1/2$. Then we have a trace map $r_m : H^s_\lambda(\mathbb{R}^n) \to \oplus_{j=0}^{m-1} H^{s-j-1/2}_\lambda(\mathbb{R}^{n-1})$.*

(ii) For any $s \in \mathbb{R}$ there exists an extension operator $E_{m,\lambda} : \oplus_{j=0}^{m-1} H^{s-j-1/2}_\lambda(\mathbb{R}^{n-1}) \to H^s_\lambda(\mathbb{R}^n)$ such that for $s > m + 1/2$, we have $r_m E_{m,\lambda} = \mathrm{id}$.

For a proof, see [16], [17].

A Resolvent Estimate

We are interested in obtaining a particular resolvent estimate for self-adjoint operators on anisotropic spaces. Here, the anisotropic space under consideration is defined on a product manifold X of the form $X = [0, 1] \times X_2$, where X_2 is a closed manifold. Consider the Hilbert space $L^2(E)$, where E is a vector bundle over X. Let B be an elliptic boundary condition, in the sense of Definition 15.24. We take $p = 2$, and we supposed B is defined for all $s_1 \geq 1$

[8] In this section, we will work exclusively with the Hilbert spaces $H^{s,2}$ and $H^{(s_1,s_2),2}$ and so we will abbreviate these spaces as simply H^s and $H^{(s_1,s_2)}$, respectively.

and s_2 in as large of a range as needed. For $s_1 = 1$ and $s_2 = 0$, we assume that the restricted operator $A_B : H^1_B(E) \to L^2(E)$ is, in addition to being Fredholm, also self-adjoint. Thus, it follows immediately from the spectral theorem that $\|(A_B - i\lambda)^{-1}\|_{Op(L^2(E))} \leq O(\lambda^{-1})$. Suppose, however, we wish to obtain a resolvent estimate on $H^{(0,s)}(E)$ instead, where $s > 0$. Then A_B is no longer an (unbounded) self-adjoint operator on $H^{(0,s)}(E)$ and we cannot apply the spectral theorem. To obtain an analogous resolvent estimate on $H^{(0,s)}(E)$, we will construct the resolvent of A_B from the λ-dependent calculus and estimate this operator directly. This resolvent will essentially be a parameter dependent version of the parametrix for A_B induced from the parametrix (15.24) for the full mapping pair (A, B). Thus, the parameter dependent calculus and Sobolev spaces developed in the previous section will come into play.

Define

$$A_{B,\lambda} := A_B - i\lambda.$$

Since A_B is self-adjoint, we know that $A^{-1}_{B,\lambda} : L^2(E) \to H^1(E)$ exists for $\lambda \in \mathbb{R} \setminus \{0\}$. In fact, we also know that $A^{-1}_{B,\lambda} : H^{(0,s)}(E) \to H^{(1,s)}(E)$ by Theorem 15.25. However, this theorem gives us no information about how the norm of $A^{-1}_{B,\lambda}$ depends on λ. The analysis which follows will show that in fact, we have

$$A^{-1}_{B,\lambda} : H^{(0,s)}_\lambda(E) \to H^{(1,s)}_\lambda(E), \tag{15.41}$$

where on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we define

$$H^{(s_1,s_2)}_\lambda(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^{(s_1,s_2)}_\lambda} = \| \langle(\xi,\lambda)\rangle^{s_1} \left\langle \xi^{(2)} \right\rangle^{s_2} \hat{f}\|_{L^2} < \infty\},$$

so that we may define $H^{(s_1,s_2)}_\lambda(E)$ in the usual way. From (15.41), it follows straight from the definitions that $\|A^{-1}_{B,\lambda}\|_{Op(H^{(0,s)})} \leq O(\lambda^{-1})$.

It remains to construct the desired resolvent operator $A^{-1}_{B,\lambda}$ and prove that it satisfies (15.41). To facilitate this, we first define operators given by certain parameter-dependent Poisson type kernels. Recall that

$$\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$$

denotes the upper half-space in $\mathbb{R}^n$.

Definition 15.29 For $m \in \mathbb{R}$, consider the space of all functions $k(x', x_n, \xi', \lambda) \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \Gamma)$ such that

$$\sup_x |\partial^\beta_{x'} \partial^\alpha_{\zeta'} x^j_n \partial^l_{x_n} k(x', x_n, \xi', \lambda)| \leq C_{\alpha\beta} \left\langle(\xi',\lambda)\right\rangle^{m+1-|\alpha|-j+l},$$

for all multi-indices α,β, and integers $j, l \geq 0$, where $\zeta' = (\xi', \lambda)$. We say that k is a *Poisson kernel of order m with parameter λ.*

Our terminology is a modified from that of [18], and such operators, among others, occur naturally in the functional calculus of pseudodifferential boundary value problems. Here we could have defined a suitable notion of polyhomogeneity for our Poisson kernels, as done in [18], but as we stated in Remark 15.26, this is not necessary. See also [16] for a more complete treatment. We have the following fundamental lemma:

Lemma 15.30 *[18, Lemma A.3] Let k be a Poisson kernel of order m with parameter λ. Then the operator*

$$Op(k)f = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} k(x', x_n, \xi', \lambda)\hat{f}(\xi')d\xi'$$

defines a bounded map

$$Op(k) : H^s_\lambda(\mathbb{R}^{n-1}) \to H^{s-m-1/2}_\lambda(\mathbb{R}^n_+)$$

for all $s \in \mathbb{R}$.

Next, we have the following important result. Let X be any compact manifold with boundary and let A be a first order elliptic formally self-adjoint differential operator on a vector bundle E over X. Let $\tilde{X}$ be any closed manifold extending X, and let $\tilde{E}$ be any extension of the vector bundle E to $\tilde{X}$. The below lemma tells us that for sufficiently large λ, the family of operators

$$A_\lambda := A - i\lambda$$

has invertible extensions to all of $\tilde{E}$. For any vector bundle V, let $OS^m_{sphg}(V)$ denote the space of space of strongly polyhomogeneous mth order pseudodifferential operators acting on sections of V.

Lemma 15.31 *[17] There exists a $\lambda_0 > 0$ such that for $|\lambda| > \lambda_0$, we have the following:*

(i) There exists an extension of A_λ to an invertible elliptic operator $\tilde{A}_\lambda \in OS^1_{sphg}(\tilde{E})$. Consequently, there exists an operator $Q_\lambda \in OS^{-1}_{sphg}(\tilde{E})$ such that $Q_\lambda A_\lambda = \mathrm{id}$ on $\tilde{X}$.

(ii) The operator A_λ on X has a Calderon projection P^+_λ that is an element of $OS^0_{sphg}(E_{\partial X})$. Moreover, the corresponding Poisson operator P_λ is of the form $Op(p_\lambda)$ where p_λ is a Poisson kernel of order -1 with parameter λ. Consequently, the maps

$$P^+_\lambda : H^s_\lambda(E_{\partial X}) \to H^s_\lambda(E_{\partial X})$$
$$P_\lambda : H^s_\lambda(E_{\partial X}) \to H^{s+1/2}_\lambda(E)$$

are bounded for all $s \in \mathbb{R}$.

We return to the relevant case where $X = X_1 \times X_2$ is a product manifold, with $\partial X = \partial X_1 \times X_2$. Let us see how Lemma 15.31 allows us to construct a resolvent for A_B. A key step is that because we can construct the "invertible double" $\tilde{A}_\lambda$, that is, an invertible extension of A_λ, we can construct a parametrix for $A_{B,\lambda}$ without any smoothing error terms, i.e., we get an honest inverse. Proceeding as before in (15.18), we have the formula

$$A_\lambda u^0 = (A_\lambda u)^0 + r^* Jru.$$

For $\lambda > \lambda_0$, by Lemma 15.31 we can apply Q_λ to both sides, where we regard u^0 as an element of $\tilde{X}$, and upon restricting back to $X \subset \tilde{X}$, we obtain the *exact* formula

$$u = Q_\lambda(A_\lambda u)^0 + Q_\lambda r^* J r u. \tag{15.42}$$

The second term in fact is exactly the operator P_λ, as given by Lemma 15.31 applied to ru. Starting from this ansatz, we can proceed to construct $A_{B,\lambda}^{-1}$ and attempt to show that it satisfies (15.41).

Proving (15.41) turns out to be remarkably technical, however, and we will restrict ourselves to the following situation, which will be general enough for our needs in Part III. We consider the case when $X_1 = [0,1]$, so that $X = [0,1] \times X_2$, and we let A be a Dirac operator on E. Letting t denote the coordinate on $[0,1]$, it follows that we can decompose the operator A as

$$A = J_t(\frac{d}{dt} + \mathsf{B}_t + \mathsf{C}_t)$$

in accordance with (3.62).

Thus, this means that J_t, B_t, and C_t are t-dependent operators on $\Gamma(E_{X_2})$, where J_t is a skew-symmetric bundle automorphism, B_t is a first order elliptic self-adjoint operator, and C_t is a zeroth order bundle endomorphism. Since $\partial X = (\{0\} \times X_2) \coprod (\{1\} \times X_2)$, let $J = J_0 \oplus -J_1$ and $\mathsf{B} = B_0 \oplus -B_1$ be the associated operators acting on boundary[9]. Since A is a Dirac operator, we have the crucial property that

$$J\mathsf{B} = -\mathsf{B}J.$$

This added algebraic structure allows us to prove the desired resolvent estimate (15.41) for the operator $A_{B,\lambda}$ for certain pseudodifferential boundary conditions B. Specifically, we prove the following theorem and Corollary 15.34, which we need for Theorem 11.7 in Part III.

Theorem 15.32 *Let A be a Dirac operator on a Hermitian vector bundle E over $X = X_1 \times X_2$, where $X_1 = [0,1]$ and X_2 is a closed manifold. Let B be a pseudodifferential projection on $\Gamma(E_{\partial X})$ satisfying the following:*

- *(i) the operator $A_B : H^1_B(E) \to L^2(E)$ is self-adjoint and Fredholm;*
- *(ii) for every $(x,\xi) \in T^*\partial X$ with $|\xi| \geq 1$, the subspace $\ker \sigma(B)(x,\xi)$ of E_x is a Lagrangian subspace with respect to the symplectic form $\mathrm{Re}\,(\cdot, J\cdot)$ on E_x, where $(\cdot,\cdot)$ is the Hermitian inner product on E_x. (Here, we choose $\sigma(B)(x,\xi)$ so that it is homogeneous of degree zero in ξ for $|\xi| \geq 1$.)*

Then the resolvent $R_\lambda = (A_B - i\lambda)^{-1}$, $\lambda \in \mathbb{R} \setminus \{0\}$, satisfies (15.41) for all $0 \leq s \leq 1$. In particular, we have

$$\|R_\lambda\|_{Op(H^{(0,s)}(E))} \leq O(|\lambda|^{-1}). \tag{15.43}$$

[9] In (3.62), we regard t as the inward normal coordinate relative to the boundary. Thus, when we consider the boundary operator B_0 on $\{0\} \times X_2$, we can take t as the coordinate on $[0,1]$, but when we consider the boundary operator B_1, we really should replace t with $\tilde{t} := 1 - t$ and work with respect to the $\tilde{t}$ coordinate in the neighborhood of $\{1\} \times X_2$. To rectify the situation, we instead work with the coordinate t for both boundary components but compensate by changing the sign of J_1, B_1, and C_1.

Proof Let λ_0 be as in Lemma 15.31, and let $\mathcal{U} = \ker B \subseteq H^{1/2}(E_{\partial X})$. Observe that $\mathcal{U}$ and $r(\ker A_\lambda)$ are transverse for all $\lambda \neq 0$. Indeed, because B defines a self-adjoint boundary condition, if $u \in H^1_B(E)$, then

$$\|A_\lambda u\|^2_{L^2(E)} = \|Au\|^2_{L^2(E)} + |\lambda|^2 \|u\|^2_{L^2(E)},$$

which is zero if and only if $u = 0$. Thus, $A_B - i\lambda$ has no kernel, which means $\mathcal{U}$ and $r(\ker A_\lambda)$ have trivial intersection. Similarly, by considering the adjoint of $A_B - i\lambda$, which is $A_B + i\lambda$, we can conclude that $\mathcal{U}$ and $r(\ker A_\lambda)$ also span $H^{1/2}(E_{\partial X})$.

Thus for $\lambda > \lambda_0$, we have two complementary projections $\Pi_{\mathrm{im}\, P^+_\lambda, \mathcal{U}}$ and $\Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda}$ on $H^{1/2}(E_{\partial X})$, with the range and kernel of $\Pi_{\mathrm{im}\, P^+_\lambda, \mathcal{U}}$ being $\mathrm{im}\, P^+_\lambda$ and $\mathcal{U}$, respectively, and vice versa for the projection $\Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda}$. For $u \in H^1_B(E)$, we have $ru \in \mathcal{U}$, so that from (15.42), we have

$$u = Q_\lambda (A_\lambda u)^0 + Q_\lambda r^* J \Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda} ru, \qquad u \in H^1_B(E). \tag{15.44}$$

We want the right-hand side to be in terms of $A_\lambda u$ only, in which case, we can substitute the expression (15.44) into the last occurrence of u in (15.44) to obtain the identity

$$u = Q_\lambda (A_\lambda u)^0 + Q_\lambda r^* J \Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda} r(Q_\lambda (A_\lambda u)^0 + Q_\lambda r^* J \Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda} u), \tag{15.45}$$

$$= Q_\lambda (A_\lambda u)^0 + P_\lambda \Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda} r Q_\lambda (A_\lambda u)^0. \tag{15.46}$$

Here, we used the fact that $rQ_\lambda r^* J = rP_\lambda = P^+_\lambda$, whose image is annihilated by $\Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda}$. Altogether, the expression on the right-hand side of (15.46) defines for us the inverse of the operator $A_{B,\lambda}$ for $|\lambda| > \lambda_0$, i.e.

$$R_\lambda = (A_B - i\lambda)^{-1} = r_X Q_\lambda E_0 + P_\lambda \Pi_{\mathcal{U}, \mathrm{im}\, P^+_\lambda} r Q_\lambda E_0. \tag{15.47}$$

Here E_0 denotes the extension by zero operator from X to $\tilde{X}$ and r_X is the restriction operator from $\tilde{X}$ to X.

We now show that $\|R_\lambda\|_{Op(H^{0,s}(E))} \leq O(\lambda^{-1})$ using the above expression for the resolvent. Let D_s be an invertible (pseudodifferential) elliptic operator of order s on X_2, with principal symbol a scalar endomorphism everywhere. Since $D_s : H^{(0,s)}(E) \to L^2(E)$ is an isomorphism for all $s \in \mathbb{R}$, to prove the desired estimate, it suffices to show that

$$\|D_s R_\lambda D_{-s}\|_{Op(L^2(X_1 \times X_2))} \leq O(\lambda^{-1}). \tag{15.48}$$

From (15.47), the first term of $D_s R_\lambda D_{-s}$ is $D_s r_X Q_\lambda E_0 D_{-s} = r_+ D_s Q_\lambda D_{-s} E_0$. Regard $D_s Q_\lambda D_{-s}$ as an anisotropic type pseudodifferential operator on the vector bundle $\tilde{E}$ over $\tilde{X} = \tilde{X}_1 \times X_2$, where $\tilde{X}_1 = S^1 \supset [0,1]$. Lemma 15.3 and the fact that $Q_\lambda \in OS^{-1}_{sphg}$ implies that the symbol $a = a(x,\xi)$ of $D_s Q_\lambda D_{-s}$ in local coordinates (x,ξ) on $T^*(X_1 \times X_2)$ satisfies

$$\sup_{x,\xi} |\partial^{\alpha^{(1)}}_{\xi^{(1)}} \partial^{\alpha^{(2)}}_{\xi^{(2)}} \partial^\beta_x a| \leq C_{\alpha\beta} \langle (\xi,\lambda) \rangle^{-1-|\alpha^{(1)}|} \left\langle \xi^{(2)} \right\rangle^{-|\alpha^{(2)}|}.$$

Thus, the symbol $\langle (\xi,\lambda) \rangle\, a$ belongs to $S^{(0,0)}$ uniformly in λ. This is equivalent to the mapping property

$$D_sQ_\lambda D_{-s} : L^2(\tilde{E}) \to H^1_\lambda(\tilde{E}) \tag{15.49}$$

Since trivially, the maps $E_0 : L^2(X_1 \times X_2) \to L^2(\tilde{X}_1 \times X_2)$ and $r_X : H^1_\lambda(\tilde{X}_1 \times X_2) \to H^1_\lambda(X_1 \times X_2)$ are bounded, and the inclusion map $H^1_\lambda(X_1 \times X_2) \hookrightarrow L^2(X_1 \times X_2)$ has norm bounded by $O(\lambda^{-1})$, it follows that

$$\|D_sr_XQ_\lambda E_0D_{-s}\|_{Op(L^2(E))} \leq O(\lambda^{-1}). \tag{15.50}$$

It remains to show that we have a similar estimate for the remaining term

$$D_sP_\lambda\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}rQ_\lambda E_0D_{-s}$$

of $D_sR_\lambda D_{-s}$. We can factor the above map as

$$(D_sP_\lambda D_{-s}) \circ (D_s\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}D_{-s}) \circ r(D_{-s}r_+Q_\lambda E_0D_{-s})$$

By the above step and Lemma 15.28, we have

$$r(D_sr_+Q_\lambda E_0D_{-s}) : L^2(E) \to H^{1/2}_\lambda(E_{\partial X}).$$

Likewise, since $P_\lambda : H^t_\lambda(E_{\partial X}) \to H^{t+1/2}_\lambda(E)$ for all $t \geq 0$, one can show that $D_sP_\lambda D_{-s} : H^t_\lambda(E_{\partial X}) \to H^{t+1/2}_\lambda(E)$ as well. The argument is the similar as to that used to establish (15.49). Namely, one can investigate the operator kernel of $D_sP_\lambda D^{-s}$, and see that it is given by a Poisson kernel of order -1 with parameter λ. One then applies Lemma 15.30. In particular, $D_sP_\lambda D_{-s} : H^{1/2}_\lambda(E_{\partial X}) \to H^1_\lambda(E)$.

Thus, to prove the theorem, it remains to estimate the second term $D_s\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}D_{-s}$ and establish the following:

Claim: The operator $D_s\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}D_{-s}$ is bounded on $H^{1/2}_\lambda(E_{\partial X})$.

We first start with the case $s = 0$. Let $P^-_\lambda = 1 - P^+_\lambda$ and let $E_\lambda = E_{0,\lambda}$ be as in Lemma 15.28. Given any $v \in H^{1/2}_\lambda(\partial X)$, define the following extension operator

$$\begin{aligned}\tilde{E}_\lambda : H^{1/2}_\lambda(E_{\partial X}) &\to H^1_\lambda(E)\\ v &\mapsto P_\lambda v + E_\lambda P^-_\lambda v.\end{aligned}$$

We have $rE_\lambda v = (P^+_\lambda + P^-_\lambda)v = v$, and so by Lemma 15.28,

$$\|v\|_{H^{1/2}_\lambda(\partial X)} \leq C\|\tilde{E}_\lambda v\|_{H^1_\lambda(X)} \tag{15.51}$$

for all v. Here, C denotes a constant independent of λ. (In what follows the precise value of C is immaterial and may change from line to line.) On the other hand, for $u \in H^1_B(E)$, the fact that A_B is elliptic implies that

$$\|u\|^2_{H^1_\lambda(E)} \leq C(\|Au\|^2_{L^2(E)} + \|u\|^2_{L^2(E)}).$$

Since $\|u\|^2_{H^1_\lambda(E)} \sim \|u\|^2_{H^1(E)} + |\lambda|^2\|u\|^2_{L^2(E)}$, then by the self-adjointness of A_B,

$$\|u\|^2_{H^1_\lambda(E)} \leq C\|A_\lambda u\|^2_{L^2(E)}. \tag{15.52}$$

Thus, for $v \in \mathcal{U}$, we have

$$\begin{aligned}\|\tilde{E}_\lambda v\|_{H^1_\lambda(E)} &\leq C\|A_\lambda \tilde{E}_\lambda v\|_{L^2(E)} \\ &= C\|A_\lambda E_\lambda P^-_\lambda v\|_{L^2(E)} \\ &\leq C'\|E_\lambda P^-_\lambda v\|_{H^1_\lambda(E)} \\ &\leq C''\|P^-_\lambda v\|_{H^{1/2}_\lambda(E_{\partial X})}.\end{aligned} \tag{15.53}$$

From (15.51) and (15.53), we have

$$\|v\|_{H^{1/2}_\lambda(E_{\partial X})} \leq C''\|P^-_\lambda v\|_{H^{1/2}_\lambda(E)}, \qquad v \in \mathcal{U}. \tag{15.54}$$

Let us see what the inequality (15.54) tells us. First, observe that we have the decompositions

$$H^{1/2}_\lambda(\partial X) \cong \operatorname{im} P^+_\lambda \oplus \operatorname{im} P^-_\lambda \tag{15.55}$$

$$H^{1/2}_\lambda(\partial X) \cong \mathcal{U} \oplus J\mathcal{U}. \tag{15.56}$$

Here, J is the boundary endomorphism on $E_{\partial X}$ defined by the Green's formula for A (cf. Proposition 15.18). The first isomorphism follows because the $P^\pm_\lambda$ are complementary projections. Moreover, since $P^\pm_\lambda \in OS^0_{sphg}$, the isomorphism (15.55) and its inverse have operator norm bounded uniformly in λ. Likewise, for (15.56), the space $\mathcal{U}$ is by definition the image of the projection $1 - B$. Since A_B is self-adjoint, $\mathcal{U}$ is a Lagrangian subspace of $H^{1/2}(X_2)$ and so $J\mathcal{U}$ is complementary to $\mathcal{U}$ in $H^{1/2}(X_2)$ (it is the L^2 orthogonal complement of $\mathcal{U}$). Since the operator norm of an element of OS^0 on λ-dependent Sobolev spaces is uniform in λ, it follows that the isomorphism (15.56) and its inverse have operator norm uniform in λ. Combining the above decompositions with the inequality (15.54), we have that

$$H^{1/2}_\lambda(X_2) \cong \mathcal{U}^{1/2} \oplus \operatorname{im} P^{+,1/2}_\lambda \tag{15.57}$$

with the isomorphism and its inverse uniform in λ. This is equivalent to the assertion made in our claim, for $s = 0$.

The case $s > 0$ requires a much more delicate analysis. In this case, we need estimates on the symbol of the projection $\Pi_{\mathcal{U}, \operatorname{im} P^+_\lambda}$, and it is here that we need the additional hypotheses involving the symbol of B (for $s = 0$, we only used self-adjointness of the boundary condition). Observe first of all that $\Pi_{\mathcal{U}, \operatorname{im} P^+_\lambda}$ is indeed pseudodifferential since its range and kernel are the range of pseudodifferential projections. Explicitly, if we let B^- denote the orthogonal projection onto $\ker B = \mathcal{U}$, observe that the pseudodifferential operator

$$T_\lambda = B^- P^-_\lambda + P^+_\lambda$$

is invertible on $H^{1/2}_\lambda(E_{\partial X})$ for every $\lambda > 0$ by the preceding analysis. (Since T_λ is pseudodifferential, note this implies that its inverse is bounded on $H^s(E_{\partial X})$ for all s). The projection $\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}$ is then simply given by

$$\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda} = T_\lambda^{-1} B^- P^-_\lambda + P^+_\lambda. \tag{15.58}$$

Nevertheless, the operator $\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}$ is still somewhat mysterious because of the presence of the term T_λ^{-1}. Indeed, we know from (15.57) that T_λ^{-1} is bounded on $H^{1/2}_\lambda(E_{\partial E})$, but it appears that we do not have any further information on its boundedness on other function spaces *uniformly* in λ. To get around this, we will approximate $\Pi_{\mathcal{U},\mathrm{im}\,P^+_\lambda}$ by a pseudodifferential operator whose dependence on λ we have complete control over. To do so, we need the following lemma:

Lemma 15.33 *Let B be a boundary condition given by a pseudodifferential projection as above. For every $(x,\xi) \in T^*(\partial X)$ with $|\xi| \geq 1$ and for all $\lambda \in \mathbb{R}$, we have the following:*

(i) The positive eigenspace of $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$ is complementary to $\ker \sigma(B)(x,\xi)$ in E_x.

(ii) Consider the projection $\tilde{\pi}(x,\xi,\lambda)$ on E_x whose image is $\ker \sigma(B)(x,\xi)$ and whose kernel is the positive eigenspace of $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$. Then $\tilde{\pi}(x,\xi,\lambda)$ is weakly homogeneous and the norm of the symbol $\tilde{\pi}(x,\xi,\lambda)$ is uniformly bounded in $|\xi| \geq 1$ and $\lambda \in \mathbb{R}$.

Let us see how this lemma allows us to prove the claim. Because of the properties of symbol $\tilde{\pi}(x,\xi,\lambda)$ as above, we can extend $\tilde{\pi}(x,\xi,\lambda)$, defined initially only on $|\xi| \geq 1$, to all of $(T^*(\partial X) \times \mathbb{R}$ in such a way that the norm of the symbol (with respect to the cotangent variables) is uniformly bounded in λ. Indeed, we can just smoothly extend $\tilde{\pi}(x,\xi,\lambda)$ to zero inside $|\xi| \leq 1/2$, and this extension can be done in such a way that the symbol norm of the resulting $\tilde{\pi}(x,\xi,\lambda)$ depends uniformly on λ because of (ii) in the above lemma.

So consider our resulting weakly homogeneous symbol $\tilde{\pi}(x,\xi,\lambda)$ defined on all of $T^*(\partial X) \times \mathbb{R}$. Define $\tilde{\Pi}_{\mathcal{U},P^+_\lambda}$ to be any weakly homogeneous pseudodifferential operator quantized from the symbol $\tilde{\pi}(x,\xi,\lambda)$ by use of a partition of unity. Namely, if we let $\{\varphi_i\}$ denote a partition of unity such that $\operatorname{supp}\varphi_i \cup \operatorname{supp}\varphi_j$ lies within a coordinate patch, for any i and j, define

$$\tilde{\Pi}_{\mathcal{U},P^+_\lambda} = \sum_{i,j} \varphi_i Op_\ell(\tilde{\pi}) \varphi_j, \tag{15.59}$$

where the left quantization is done with respect to some chart containing $\operatorname{supp}\varphi_i \cup \operatorname{supp}\varphi_j$. Since the symbol $\tilde{\pi}$ lies in OS^0 uniformly in λ, it follows that the operator norm of $\tilde{\Pi}_{\mathcal{U},P^+_\lambda}$ on *any* Sobolev space is bounded uniformly in λ.

We now compare the pseudodifferential operator $\tilde{\Pi}_{\mathcal{U},P^+_\lambda}$ with the projection $\Pi_{\mathcal{U},P^+_\lambda}$. In the below, let $Op(-1)$ denote any λ-dependent pseudodifferential operator which belongs to OS^{-1} uniformly in λ. The precise value of $Op(-1)$ is immaterial and may change from line to line. Then we have for $v \in \mathcal{U}$ that

$$\begin{aligned}\tilde{\Pi}_{\mathcal{U},P_\lambda^+} v &= B^- v + (\tilde{\Pi}_{\mathcal{U},P_\lambda^+} - B^-)v \\ &= v + (\tilde{\Pi}_{\mathcal{U},P_\lambda^+} B^- - B^-)v \\ &= v + Op(-1)v.\end{aligned}$$

The last line follows from the fact that $\tilde{\Pi}_{\mathcal{U},P_\lambda^+} B^-$ and B^- have the same principal symbols and the uniformity property of $\tilde{\Pi}_{\mathcal{U},P_\lambda^+}$. Likewise, we have for $v^+ \in \operatorname{im} P_\lambda^+$ that

$$\tilde{\Pi}_{\mathcal{U},P_\lambda^+} v^+ = \tilde{\Pi}_{\mathcal{U},P_\lambda^+} P_\lambda^+ v^+ = Op(-1)v$$

since $\operatorname{im}\sigma(P_\lambda^+) \subseteq \ker \sigma(\tilde{\Pi}_{\mathcal{U},P_\lambda^+})$. Since $\mathcal{U}$ and P_λ^+ span $H_\lambda^{1/2}(E_{\partial X})$, the above computations show that

$$\tilde{\Pi}_{\mathcal{U},P_\lambda^+} = \Pi_{\mathcal{U},P_\lambda^+} + Op(-1)\Pi_{\mathcal{U},P_\lambda^+} + Op(-1). \tag{15.60}$$

Thus, one now has

$$D^s \Pi_{\mathcal{U},P_\lambda^+} D^{-s} = D^s \tilde{\Pi}_{\mathcal{U},P_\lambda^+} D^{-s} + D^s Op(-1)\Pi_{\mathcal{U},P_\lambda^+} D^{-s} + D^s Op(-1) D^{-s}. \tag{15.61}$$

We want to show that the operators on the right-hand side of the above are bounded on $H_\lambda^{1/2}(E_{\partial X})$. The first term is bounded on $H^{1/2}(E_{\partial X})$ because $\tilde{\Pi}_{\mathcal{U},P_\lambda^+}$ is defined by (15.59) and $\tilde{\pi}$ belongs to OS^0 uniformly in λ. The same argument applies to show that the term $D^s Op(-1) D^{-s}$ is bounded on $H_\lambda^{1/2}(E_{\partial X})$. The mysterious term is $D^s Op(-1)\Pi_{\mathcal{U},P_\lambda^+} D^{-s}$, since we only know that $\Pi_{\mathcal{U},P_\lambda^+}$, though pseudodifferential, is bounded on $H_\lambda^{1/2}(E_{\partial X})$. Essentially, we do not have any control over the total symbol of $\Pi_{\mathcal{U},P_\lambda^+}$ and hence cannot deduce its boundedness on other Sobolev spaces. However, we do the simplest thing possible to bound the last term of (15.61). We have that the λ-independent operator D^{-s} is a bounded operator on $H_\lambda^{1/2}(E_{\partial X})$, since $s \geq 0$. We also have that $D^s Op(-1)$ belongs to OS^0 uniformly in λ for $s \leq 1$ and hence is bounded on $H_\lambda^{1/2}(E_{\partial X})$ for $s \leq 1$. It now follows that the composite operator $D^s Op(-1)\Pi_{\mathcal{U},P_\lambda^+} D^{-s}$ is bounded on $H_\lambda^{1/2}(E_{\partial X})$. This completes the proof that $D^s \Pi_{\mathcal{U},P_\lambda^+} D^{-s}$ is bounded on $H_\lambda^{1/2}(E_{\partial X})$, which proves the claim and hence the theorem. □

We now prove Lemma 15.33 to complete the proof of Theorem 15.32.

Proof of Lemma 15.33: (i) For any fixed $x \in \partial X$, the symbols $\sigma(B)(x,\xi)$ and $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$ are endomorphisms of E_x. Furthermore, viewing $\sigma(B)(x,\xi)$ as being a constant function of λ, then $\sigma(B)(x,\xi)$ is weakly homogeneous while $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$ is strongly homogeneous. Thus, to compare the spectrum of these two matrices as (ξ,λ) varies, the above homogeneity properties imply that we need only consider these matrices on the cylinder $|\xi| = 1$ in $(\xi,\lambda) \in T_x^* \times \mathbb{R}$.

Let $E_x^{\pm}(\xi,\lambda)$ denote the positive (negative) eigenspace of $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$. Observe that $E_x^{\pm}(\xi,\lambda)$ are complementary subspaces of E_x for all $|\xi| = 1$ and λ. Indeed, for $\lambda = 0$, this follows from the fact that $E_x = E^+(x,\xi,\lambda) \oplus E^-(x,\xi,\lambda)$ and J is an isomorphism that interchanges the positive and negative eigenspaces. This follows from B being elliptic and the relation $\mathsf{B}J = -J\mathsf{B}$. For all $\lambda \neq 0$, then $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$ remains an invertible self-adjoint elliptic operator, which means that the dimensions of $E^{\pm}(x,\xi,\lambda)$ remain constant as λ varies. Thus, the $E_x^{\pm}(\xi,\lambda)$ are complementary subspaces of E_x for all λ.

We now proceed with the proof of the lemma. For $\lambda = 0$, we have that $\ker\sigma(B)(x,\xi)$ and $E_x^+(\xi,\lambda)$ intersect trivially since B is an elliptic boundary condition. Since in addition, $\ker\sigma(B)(x,\xi)$ and $E_x^+(\xi,\lambda)$ are Lagrangian subspaces of E_x, they must in fact be complementary. (The space $E_x^+(\xi,\lambda)$ is Lagrangian since the spaces $E_x^{\pm}(\xi,\lambda)$ are orthogonal and interchanged by J.)

For $\lambda \neq 0$, we consider the following computation for any two elements $u, v \in E_x^+(\xi,\lambda)$. Write $\sigma(\mathsf{B}) = \sigma(\mathsf{B})(x,\xi)$ for shorthand. Since $u, v \in E_x^+(\xi,\lambda)$, we have $\sigma(\mathsf{B} + i\lambda J)u = \mu u$ and $\sigma(\mathsf{B} + i\lambda J)v = \mu' v$ for some $\mu, \mu' > 0$. It follows that

$$\begin{aligned}
\mathrm{Re}\,(u, Jv) &= \frac{1}{\mu'}\mathrm{Re}\,(u, J(\sigma(\mathsf{B}) + i\lambda J)v) \\
&= \frac{1}{\mu'}\mathrm{Re}\,((J(\sigma(\mathsf{B}) - i\lambda J)u, v) \\
&= \frac{1}{\mu'}\mathrm{Re}\,((J(\sigma(\mathsf{B}) + i\lambda J - 2i\lambda J)u, v) \\
&= \frac{\mu}{\mu'}\mathrm{Re}\,(Ju, v) + \frac{1}{\mu'}\mathrm{Re}\,(2i\lambda u, v)
\end{aligned}$$

In the above, we use that $J\sigma(\mathsf{B})$ is self-adjoint, $J^* = -J$, and $J^2 = -1$. Rearranging, this implies

$$\mathrm{Re}\,(u, Jv) = -(\mu + \mu')^{-1}\mathrm{Re}\,(u, 2i\lambda v). \tag{15.62}$$

Suppose $w \in \ker\sigma(B) \cap E_x^+(\xi,\lambda)$ is nonzero. Then since $\ker\sigma(B)$ is both a Lagrangian subspace of E and a complex vector space, we must have $\mathrm{Re}\,(w, J\alpha w) = 0$ for all $\alpha \in \mathbb{C}$. For nonzero $\alpha \in i\mathbb{R}$ however, setting $v = \alpha u$ contradicts (15.62). It follows that $\ker\sigma(B) \cap E_x^+(\xi,\lambda) = 0$ and hence these two spaces must be complementary.

(ii) By (i), since $\ker\sigma(B)(x,\xi)$ and $E_x^+(\xi,\lambda)$ are complementary for $|\xi| \geq 1$ and $\lambda \in \mathbb{R}$, the projection $\tilde{\pi}(x,\xi,\lambda)$ is well-defined and uniquely defined. It is clearly weakly homogeneous since both $\ker\sigma(B)(x,\xi)$ and $\sigma(\mathsf{B} + i\lambda J)(x,\xi)$ are. It remains to establish the uniformity statement, which by weak homogeneity, we need only establish on the cylinder $|\xi| = 1$ in $T_x^*(\partial X)$, $\lambda \in \mathbb{R}$. This amounts to showing that that the "distance" between $\ker\sigma(B)(x,\xi)$ and $E_x^+(\xi,\lambda)$ is uniformly bounded along the cylinder (where to define distance, one can pick any metric on $Gr = Gr(E_x, \dim(E_x)/2)$, the Grassmanian of E_x consisting of subspaces of half-dimension). Observe that for any $\lambda^* > 0$, we have a uniform estimate on the norm of $\tilde{\pi}(x,\xi,\lambda)$ on E_x since the set $|\xi| = 1$ and $|\lambda| < \lambda^*$ is compact. Thus, the essential task is to get a uniform estimate on the norm of $\tilde{\pi}(x,\xi,\lambda)$ on the non-compact set $|\lambda| > \lambda^*$. Observe however that as $\lambda \to \infty$, the matrix $\sigma(\lambda^{-1}\mathsf{B} + iJ)(x,\xi)$ converges to iJ uniformly on the compact set $|\xi| = 1$ in $T_x^*(\partial X)$. In other words, $E_x^+(\xi,\lambda)$ converges to $E_x^+(0,1)$ (in the Grassmanian Gr). The same analysis in (i) shows that $E_x^+(0,1)$ is complementary to $\ker\sigma(B)(x,\xi)$ for all ξ such that $|\xi| = 1$. By continuity, it follows that for every ξ with

$|\xi| = 1$, the set $\{E^+(x,\xi,\lambda)\}_{\lambda>\lambda^*}$ is uniformly separated (i.e. has a positive distance) from the point $\ker\sigma(B)(x,\xi)$ in Gr. Throwing in the compact set $|\lambda| < \lambda^*$, we see that in fact $\{E^+(x,\xi,\lambda)\}_{\lambda\in\mathbb{R}}$ has a positive distance from $\ker\sigma(B)(x,\xi)$ in Gr. Letting (x,ξ) vary over the compact set $|\xi| = 1$ in $T^*(\partial X)$, we deduce that there is a fixed positive distance between $E^+(x,\xi,\lambda)$ and $\ker\sigma(B)(x,\xi,\lambda)$ for all $(x,\xi) \in T^*(\partial X) \in \mathbb{R}$ with $|\xi| \geq 1$. This proves the uniformity statement. □

The proof of the Theorem 15.32 shows that the following slight generalization holds:

Corollary 15.34 *Let B' be a projection such that there exists a pseudodifferential projection B satisfying the hypotheses of Theorem 15.32 and $B' - B$ is an operator which is smoothing of order one, i.e. $(B' - B) : H^t(E_{\partial X}) \to H^{t+1/2}(E_{\partial X})$ for all $t \geq 1/2$. Then the resolvent $R_\lambda = (A_{B'} - i\lambda)^{-1}$, $\lambda \in \mathbb{R}\setminus\{0\}$, satisfies (15.43) for all $0 \leq s \leq 1$.*

16 Vector Valued Cauchy Riemann Equations

In this section, we state a modified version of the results of [52], both to strengthen them for our needs and also to correct some subtle errors. Specifically, we need to make use of the elliptic estimates obtained in [52] for Banach space valued (i.e. vector valued) Cauchy Riemann equations with totally real boundary conditions. Namely, consider the following situation. We have a Banach space X endowed with a complex structure, i.e., an endomorphism $J : X \to X$ such that $J^2 = -\mathrm{id}$. A subspace $Y \subset X$ is said to be totally real if $X \cong Y \oplus JY$. A submanifold $\mathfrak{L} \subset X$ is said to be totally real if each of its tangent spaces is a totally real subspace of X. In particular, for the situation that concerns us, if X is a symplectic Banach space which is densely contained within a Hilbert space H, and $J : H \to H$ is a complex structure which preserves X, then Lagrangian subspaces and Lagrangian submanifolds of X are all totally real. For simplicity, we assume we are in this symplectic situation, though everything we do generalizes to the general case.

Given a Lagrangian submanifold $\mathfrak{L} \subset X$ and some $1 < p < \infty$, we assume the following hypothesis:

$(\mathrm{I})_p$ There exists a (finite dimensional) vector bundle E over some closed manifold M, such that each tangent space to $\mathfrak{L}$ is isomorphic to a closed subspace of the Banach space $L^p(E)$, the space of all L^p sections of E.

In the above hypothesis, we assume an inner product on E is chosen so that an L^p norm is defined. In [52], the case where E is a trivial bundle is considered, but one can see from the methods there that the more general case can be easily deduced from this latter case.

The next hypothesis, which is omitted from [52], is one concerning analyticity of the submanifold $\mathfrak{L}$. Recall from Definition 21.1 the notion of an analytic map between two Banach spaces. From this, we can define the notion of an analytic Banach submanifold:

Definition 16.1 Let X be a Banach space. An *analytic Banach submanifold* M of X is a subspace of X (as a topological space) that satisfies the following. There exists a closed Banach subspace $Z \subset X$ such that at every point $u \in M$, there exists an open set V in X containing u and an analytic diffeomorphism Φ from V onto an open neighborhood of 0 in X such that $\Phi(V \cap M) = \Phi(V) \cap Z$. We say that M is modeled on the Banach space Z.

We have the following additional hypothesis for our Lagrangian:

(II) The Lagrangian submanifold $\mathfrak{L} \subset X$ is an analytic Banach submanifold of X.

Without hypothesis (II), Theorem 1.2 of [52] is incorrect as stated. We explain what modifications need to be made at the end of this section.

Given the above hypotheses, we consider the following situation. Let $\Omega \subset \mathbb{H}$ be a bounded open subset of the half-space

$$\mathbb{H} = \{(t, v) \in \mathbb{R}^2 : v \geq 0\}.$$

Given any Banach space X and $1 \leq p \leq \infty$, consider the vector-valued Sobolev spaces $W^{k,p}(\Omega, X)$, defined as in Definition 13.25 but with Ω replacing $\mathbb{R}^n$.

Observe that if we have a bounded multiplication map

$$W^{k,p}(\Omega) \times W^{k',p'}(\Omega) \to W^{k'',p''}(\Omega) \tag{16.1}$$

on the usual scalar valued Sobolev spaces, then this induces a bounded multiplication map

$$W^{k,p}(\Omega, \mathrm{End}(X)) \times W^{k',p'}(\Omega, X) \to W^{k'',p''}(\Omega, X). \tag{16.2}$$

Suppose we are given a Lagrangian submanifold $\mathfrak{L} \subset X$ satisfying $(\mathrm{I})_p$ and (II) for some $1 < p < \infty$. Let $u : \Omega \to X$ be a map that satisfies the boundary value problem

$$\begin{aligned} \partial_t u + J\partial_v u &= G \\ u(t,0) &\in \mathfrak{L}, \text{ for all } (t,0) \in \partial\Omega \cap \partial\mathbb{H}, \end{aligned} \tag{16.3}$$

where $G : \Omega \to X$ is some inhomogeneous term. Thus, the system (16.3) is a Cauchy-Riemann equation for the Banach space valued function u, supplemented with a Lagrangian boundary condition. In [52], the complex structure $J = J_{t,v}$ is allowed to vary with $t, v \in \Omega$. For simplicity, and since we will not need to assume otherwise, we let J be constant.

We have the following elliptic regularity theorem for the equations (16.3), which is a refined and corrected version of [52, Theorem 1.2]:

Theorem 16.2 *Fix $1 < p < \infty$, let $k \geq 1$, and let $K \subset \mathrm{int}\,\Omega$ be a compact subset. Let $\mathfrak{L} \subset X$ be a Lagrangian submanifold satisfying $(\mathrm{I})_p$ and (II).*

(i) Suppose $u \in W^{k,q'}(\Omega, X)$ solves (16.3) with $G \in W^{k,q}(\Omega, X)$, for some q and q' satisfying $p \leq q \leq q' < \infty$. Furthermore, suppose q', q, and p are such that we have bounded multiplication maps

$$W^{k-1,q'}(\Omega) \times W^{k-1,q'}(\Omega) \to W^{k-1,p}(\Omega) \tag{16.4}$$

$$W^{k-1,q'}(\Omega) \times W^{k,q}(\Omega) \to W^{k-1,p}(\Omega). \tag{16.5}$$

Then $u \in W^{k+1,p}(K, X)$.

(ii) Furthermore, let $u_0 \in C^\infty(\Omega, X)$ be such that $u_0(t,0) \in \mathfrak{L}$ for all $(t,0) \in \partial\Omega \cap \partial\mathbb{H}$. Then there exists a $\delta > 0$ depending on u_0 such that if $\|u - u_0\|_{L^\infty(\Omega,X)} < \delta$ is

sufficiently small, then

$$\|u-u_0\|_{W^{k+1,p}(K,X)} \leq C\big(\|G\|_{W^{k,q}(\Omega,X)} + \|u-u_0\|_{W^{k,q'}(\Omega,X)}\big). \tag{16.6}$$

where C is a constant bounded in terms of δ, u_0, and $\|u-u_0\|_{W^{k,q'}(\Omega,X)}$.

Let us note the main differences between Theorem 16.2 and [52, Theorem 1.2]. First, we allow for a more general range of q. In [52], only the case $q' = q$ is considered, in which case the range permissible of q is such that

$$W^{k-1,q}(\Omega) \times W^{k-1,q}(\Omega) \to W^{k-1,p}(\Omega) \tag{16.7}$$

is bounded, which is more restrictive than (16.4)–(16.5). Second, we now explain why we assume the analyticity hypothesis (II). If this is omitted, then the theorem above must be modified as follows: the constant δ appearing in Theorem 16.2(ii) a priori depends on k. While this seems innocuous, this implies that if one wishes to use (16.6) to bootstrap the regularity of u to higher and higher regularity (k increasing to infinity), one also needs u to get closer and closer to u_0. Analyticity, however, ensures that a small enough δ works for all k. In a few words, this is because analytic maps, being expressible as a power series locally near any point, satisfy good estimates for all their derivatives within a *fixed* neighborhood of any point. More precisely, for an analytic map, there is a fixed neighborhood about any point on which the kth Fréchet derivative of the map is Lipschitz for every k. This is precisely Proposition 21.3. Hence, hypothesis (II) ensures that our analytic Banach submanifold has local chart maps obeying Lipschitz estimates on fixed neighborhoods, which gives us the uniformity of δ with respect to k in Theorem 16.2.

We now give a very cursory explanation for how one modifies the proof of Theorem 1.2 of [52] to prove Theorem 16.2, since only very minor changes are needed. There are two places where modifications need to be made. The first one, as we have mentioned, is the issue with analyticity. In detail, in the proof of Theorem 1.2, an estimate of the form $\|v\|_{W^{k,q}(\Omega,X)} \leq C\|u-u_0\|_{W^{k,q}(\Omega,X)}$ is made for a certain configuration v when $\|u-u_0\|_{L^\infty(\Omega,X)} \leq \delta$. One can inspect from the assumptions made there that analyticity needs to be assumed, else δ a priori depends on k, as can be seen from the above discussion and the discussion preceding Proposition 21.3. The second modification to be made, so that we may sharpen the results of [52, Theorem 1.2], concerns refinements in Sobolev multiplication. In [52, Theorem 1.2], after equation[10] (9), one bounds $\|u-u_0\|_{W^{k+1,p}(K,X)}$ by bounding certain functions $F \in W^{k-1,p}(U,X)$ and $H \in W^{k,p}(U,X)$, where $U \subset \Omega$ is an open set containing K. However, from the definitions of these functions, one has the schematic bound

$$\|F\|_{W^{k-1,p}} + \|H\|_{W^{k,p}} \leq c\left(\|f\|_{W^{k,p}} + \|(\nabla I)f\|_{W^{k-1,p}} + \|(\nabla I)(\nabla v)\|_{W^{k-1,p}}\right) \tag{16.8}$$

for some constant c and certain configurations I, f, and v as defined in the proof. (In the above, ∇ denotes the 1-jet.) In [52], the assumption that $G \in W^{k,q}$ and $u \in W^{k,q}$ is used

[10]This equation number refers to the version of the paper appearing on January 27, 2004 at `http://arxiv.org/abs/math/0401376`.

to prove that

$$\|v\|_{W^{k,q}},\ \|I\|_{W^{k,q}} \leq C\|u-u_0\|_{W^{k,q}}$$
$$\|f\|_{W^{k,q}} \leq C(\|G\|_{W^{k,q}} + \|u-u_0\|_{W^{k,q}},$$

for some constant C depending on $\|u-u_0\|_{W^{k,q}}$. The same reasoning shows that we also have the bound

$$\|v\|_{W^{k,q'}},\ \|I\|_{W^{k,q'}} \leq C\|u-u_0\|_{W^{k,q'}}$$

for C depending on $\|u-u_0\|_{W^{k,q'}}$. Thus, from this bound on I and the bound on f, the multiplication hypotheses (16.4) and (16.5) imply the bound

$$\|F\|_{W^{k-1,p}} + \|H\|_{W^{k,p}} \leq C\big(\|G\|_{W^{k,q}} + \|u-u_0\|_{W^{k,q'}}\big). \tag{16.9}$$

Everything now follows through all the same in [52], and the above bound gives us the bound (16.6) and hence the theorem.

17 Unique Continuation

Let $A : \Gamma(E) \to \Gamma(F)$ be a smooth Dirac operator acting between sections of the Hermitian vector bundles E and F over a compact manifold X (with or without boundary). The operator A is said to obey the unique continuation property if every u that solves $Au = 0$ and which vanishes on an open subset of X vanishes identically. It is well-known that Dirac operators obey the unique continuation property. If X is a manifold with boundary, we can replace the condition that u vanish on an open set with the condition $ru = 0$, where $r = r_0$ is the restriction map to the boundary. This is because one can extend the operator A to a Dirac operator $\tilde{A}$ on an open manifold $\tilde{X}$ that contains X in its interior, and one can extend u to $\tilde{X}$ by zero outside of X. Since $\tilde{A}$ is a first order operator, then $\tilde{A}\tilde{u} = 0$ on $\tilde{X}$. Since $\tilde{u}$ vanishes on an open set, then $\tilde{u} \equiv 0$ on $\tilde{X}$ and so $u \equiv 0$ on X.

The following is a well-known general result:

Theorem 17.1 *Let X be a compact manifold with boundary, let D be a smooth Dirac operator on $\Gamma(E)$, and let V be an L^∞ multiplication operator. Then $D+V$ has the unique continuation property. More precisely, if $u \in B^{1,2}(E)$ satisfies $(D+V)u = 0$ and $ru = 0$, then $u \equiv 0$.*[11]

One application of this theorem is to show that such an operator $D+V$, acting between suitable function spaces, is surjective on a manifold with boundary. This is in contrast to when X is closed, in which case $D+V$ is only Fredholm, in which case it may have a finite dimensional cokernel. We have the following theorem:

Theorem 17.2 *Let X be a compact manifold with boundary. Let $2 \leq p < \infty$, $s > 1/p$ and*

[11] One can start with u of lower regularity than $B^{1,2}(E)$, say $L^2(E)$, since by elliptic bootstrapping, such a u will necessarily be of regularity $B^{1,2}(E)$, see the proof of 17.2 in [34].

let $D+V : B^{s,p}(E) \to B^{s-1,p}(F)$ *where* V *is a sufficiently smooth*[12] *multiplication operator. Then* $D+V$ *is surjective.*

Proof Since $D+V$ is a smooth elliptic operator, it has a right (pseudodifferential) parametrix. This shows that $D+V$ has closed range and finite dimensional cokernel. It remains to show that the cokernel is zero. There are two cases to distinguish, the cases $s>1$ and $s \le 1$. Let us deal with the latter case, with the case $s>1$ similar. Suppose $u \in (B^{s-1,p}(F))^* = B^{1-s,p'}(F)$, $p' = p/(p-1)$, belongs to the dual space of $B^{s-1,p}(F)$ and annihilates $\mathrm{im}\,(D+V) \subseteq B^{s-1,p}(F)$. We want to show that $u=0$, which combined with the fact that $\mathrm{im}\,(D+V)$ is closed means that $\mathrm{im}\,(D+V)$ is all of $B^{s-1,p}(F)$. The condition that u annihilate $\mathrm{im}\,(D+V)$ means that we have the (weak) equation $(D+V)^*u = 0$, and thus, $Du = -V^*u$ (here we think of dual operator D^* acting on the linear functional u as being the same as D, since a Dirac operator is formally self-adjoint). We have $V^*u \in B^{1-s,p'}$, since multiplication by a smooth function is bounded on all Besov spaces. By Theorem 15.19(i), we have a well-defined trace $r(u) \in B^{1-s-1/p',p'}(F_{\partial X})$. Thus, for all $v \in B^{s,p}(E)$, we have Green's formula (3.65), which tells us that

$$\begin{aligned} 0 &= (v,(D+V)u) - ((D+V^*)v,u) \\ &= \int_{\partial X} (r(v), -Jr(u)). \end{aligned} \tag{17.1}$$

The first line follows since u annihilates $\mathrm{im}\,(D+V)$ and $(D+V^*)u = 0$. The second line is well-defined since $Jr(v) \in B^{s-1/p,p}(F)$ and $B^{s-1/p,p}(F)$ is the dual space of $B^{-s+1/p,p'}(F) = B^{1-s-1/p',p'}(F)$. Since (17.1) holds for all $v \in B^{s,p}(E)$, it follows that $r(u)=0$. The system $(D+V^*)u = 0$ and $r(u)=0$ is overdetermined which means that we have an elliptic estimate for u via Theorem 15.19. That is, since $Du = -V^*u$, we have an estimate of the form

$$\|u\|_{B^{t+1,q}} \le C(\|V^*u\|_{B^{t,q}} + \|u\|_{B^{t,q}}). \tag{17.2}$$

for all t,q such that the right-hand side is finite, $t+1 > 1/q$. Since $u, V^*u \in B^{1-s,p'}(E)$ we have $u \in B^{2-s,p'}(E)$, where $2-s > 1/p'$ since $s \le 1$. Feeding this back into (17.2) and using that V is smooth, we see that we can boostrap u to any desired regularity. Thus u is smooth. (In general, for V not smooth, we want V sufficiently regular so that the above steps allow us to bootstrap to $u \in B^{1,2}(E)$). Furthermore, $r(u)=0$. We now apply Theorem 17.1 to conclude $u=0$. Thus, $D+V$ is surjective. □

In [34], we sketched the proof of some unique continuation results. These are as follows:

Theorem 17.3 *Assume (4.1) and* $s > \max(3/p, 1/2)$. *If* $(B_1,\Psi_1),(B_2,\Psi_2) \in \mathfrak{M}^{s,p}$ *are irreducible and satisfy* $r_\Sigma(B_1,\Psi_1) = r_\Sigma(B_2,\Psi_2)$*, then* (B_1,Ψ_1) *and* (B_2,Ψ_2) *are gauge equivalent on* Y.

[12]To keep the function space arithmetic simple, we suppose V is smooth in the proof in the theorem, though the necessary modifications can be made for V nonsmooth but bounded as a map between suitable function spaces, depending on s,p. What mainly needs to carry through is the bootstrapping argument in (17.2). In all applications, we will always have $V \in B^{t,p}(Y)$ where t is sufficiently large so that the statement remains true with V of this regularity class. If $s \ge 1$, one can check that $V \in L^\infty(E)$ suffices. If $s<1$, one wants V to have some regularity so that it can act via multiplication on functions of low regularity.

Theorem 17.4 *Let* $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$, $s > 3/p$. *Suppose* $(b,\psi) \in \mathcal{T}^{1,2}$ *satisfies* $\mathcal{H}_{(B,\Psi)}(b,\psi) = 0$ *and* $r_\Sigma(b,\psi) = 0$. *Then either (i)* $(b,\psi) \in \mathcal{J}^{1,2,\mathrm{loc}}_{(B,\Psi),t}$ *or else (ii)* $\Psi \equiv 0$, *and then* $\psi \equiv 0$ *and* $b \in \ker d$.

Corollary 17.5 *Let* $(B,\Psi) \in \mathfrak{M}^{s,p}(Y)$, $s > 3/p$. *Suppose* $(b,\psi) \in \mathcal{K}^{1,2}_{(B,\Psi)}$ *satisfies* $\mathcal{H}_{(B,\Psi)}(b,\psi) = 0$ *and* $r_\Sigma(b,\psi) = 0$. *Then either (i)* $(b,\psi) = 0$ *or else (ii)* $\Psi \equiv 0$ *and* $b \in H^1(Y,\Sigma; i\mathbb{R}) \cong \{a \in \Omega^1(Y; i\mathbb{R}) : da = d^*a = 0, a|_\Sigma = 0\}$.

As explained in [34], to complete the sketch of the proof requires a version of [21, Proposition 7.2.3] for lower regularity configurations. We now verify that this is the case.

Let H be a Hilbert space and $L : H \to H$ a (possibly unbounded) symmetric operator. For any open interval $I \subset \mathbb{R}$, define the vector-valued Sobolev space

$$W^{1,2}_L(I,H) = \{z \in W^{1,2}(I,H) : Lz \in L^2(I,H)\}.$$

Observe that we have the embedding $W^{1,2}_L(I,H) \hookrightarrow C^0(I,H)$.

Lemma 17.6 *[21, Lemma 7.1.3] Let* $z \in W^{1,2}_L(I,H)$ *be a solution (in the sense of* H*-valued distributions) to the equation*

$$\frac{dz}{dt} + L(t)z = f(t),$$

where $f \in C^0(I,H)$. *Suppose in addition that* $f(t)$ *satisfies*

$$\|f(t)\|_H \le \delta \|z(t)\|_H, \qquad \forall t \in I,$$

for some constant δ. *Then if* $z(t)$ *is zero for some* $t \in I$, *it follows that* z *is identically zero.*

Proof To prove the lemma, it suffices to show that if z is nonzero at any $t_0 \in I$, then z is nonzero at every point in I. In [21, Lemma 7.1.3], this method of proof is carried out under the stronger regularity assumptions on z. We will therefore establish our lemma by checking that the steps made in [21, Lemma 7.1.3] still hold under our more general hypotheses. To begin, picking $t_0 \in I$, one defines the quantity

$$l(t) = \log \|z(t)\| - \int_{t_0}^{t} \frac{\langle f(\tau), z(\tau)\rangle}{\|z(\tau)\|^2} d\tau,$$

which is a continuous function since $z(t)$ and $f(t)$ are continuous. If $z(t)$ were sufficiently smooth, then one can verify as in [21, Lemma 7.1.3], that we have

$$\dot{l}(t) = \frac{-\langle Lz, z\rangle}{\|z\|^2} \tag{17.3}$$

$$\ddot{l}(t) = \frac{1}{\|z\|^4}\left(2\|Lz\|^2\|z\|^2 - 2|\langle Lz, z\rangle|^2 - 2\langle Lz, f\rangle \|z\|^2 + 2\langle Lz, z\rangle\langle f, z\rangle - \left\langle \dot{L}z, z\right\rangle \|z\|^2\right), \tag{17.4}$$

by simply differentiating the expression for $l(t)$. In our situation, where $z \in W^{1,2}_L(I,H)$, one can no longer perform strong differentiation in t but only differentiation in the sense of distributions. Nevertheless, the expressions appearing on the right-hand side of (17.3) and (17.4) belong to $L^1(I)$, and hence correspond to well-defined distributions. It follows that the equalities (17.3) and (17.4) hold in the sense of distributions. This reasoning for this is standard: First, instead of working with $l(t)$, we mollify it, i.e., we replace $l(t)$ by $l_\epsilon(t) = (\varphi_\epsilon * l)(t)$, where φ is a smooth compactly supported function such that $\varphi_\epsilon = \epsilon^{-1}\varphi(\epsilon^{-1}t)$ approaches a delta function as $\epsilon \to 0$. Then since $l_\epsilon \to l$ as a distribution, the same is true for all the corresponding distributional derivatives. On the other hand, since l_ϵ is smooth, one can take strong derivatives of l_ϵ and then let $\epsilon \to 0$. We leave it as an exercise to the reader that upon doing this regularization procedure, one obtains the equalities (17.3) and (17.4), for $t \in I$. The rest of the argument in [21, Lemma 7.1.3] now goes through as before. Namely, simple algebraic manipulations of the above expressions yield the differential inequality

$$\ddot{l} + C_3|\dot{l}| + C_5 \geq 0,$$

where C_3 and C_5 are absolute constants. Here, the inequality holds in the sense of $L^1(I)$. It follows that if we define $u(t) = e^{-C_3 t}\dot{l}$, then u satisfies

$$\dot{u} + C_5 e^{-C_3 t} \geq 0$$

at all points where $u < 0$. This statement makes sense since u is continuous, as we have a continuous embedding $W^{1,1}(I) \hookrightarrow C^0(I)$. Thus, the function $u_- = \min(0, u)$ satisfies

$$u_-(t) \geq u_-(t_0) - C_5 e^{-C_3(t-t_0)}$$

for all $t \in I$. It follows that u is bounded from below on I, and hence so is $l(t)$. The bound on f implies

$$\log \|z(t)\| \geq l(t) - \delta|t - t_0|$$

for $t \in I$. Thus, $\log \|z(t)\|$ is bounded from below and hence $\|z(t)\|$ is bounded away from 0 for all $t \in I$. □

Appendix A

Some Additional Functional Analysis

18 Subspaces and Projections

Here, we collect some properties about projections and subspaces of Banach spaces. Given a Banach space X, a projection π is a bounded operator such that $\pi^2 = \mathrm{id}$. A subspace $U \subset X$ is complemented if there exists another closed subspace $V \subset X$ such that $X = U \oplus V$. In this case, $X \cong U \oplus V$. A closed subspace U is complemented if and only if there exists a projection π whose range is U. In this case, we have the decomposition

$$X = \operatorname{im}\pi \oplus \ker\pi.$$

Recall that any finite dimensional subspace of a Banach space is complemented. Likewise, any subspace of finite codimension is also complemented. Thus, if $Y \subset X$ has finite (co)dimension, we may always regard X/Y as a subspace of X (though unless X is a Hilbert space, there is in general no canonical embedding $X/Y \hookrightarrow X$).

The following simple lemma tells us that if a projection π restricted to a subspace $U' \subset X$ yields a Fredholm map $\pi : U' \to \operatorname{im}\pi$, then U' is essentially a graph over $\operatorname{im}\pi$. More precisely, we have the following:

Lemma 18.1 *Let $X = U \oplus V$ and let π be the projection onto the first factor. Let U' be a subspace of X and suppose $\pi : U' \to U$ is Fredholm. Then*

$$U' = \{x + Tx : x \in \pi(U')\} \oplus F, \tag{18.1}$$

where $F = \ker(\pi|_{U'})$ is finite dimensional and $T : \pi(U') \to V$ is a bounded operator. Consequently, U' is also a complemented subspace of X, in particular, it is the range of a projection.

Proof Since F is finite dimensional, it is the range of a bounded projection $\pi_F : X \to F$. Since $F \subset U'$, then $(1 - \pi_F) : U' \to U'$ maps U' into itself and its range is a complement of F in U'. It follows that $\pi : (1 - \pi_F)(U') \to \pi(U')$ is an isomorphism. Let $\bar{\pi}$ denote this isomorphism. Thus,

$$\begin{aligned} U' &= \{(\bar{\pi})^{-1}x : x \in \pi(U')\} \oplus F \\ &= \{x + (-1 + (\bar{\pi})^{-1})x : x \in \pi(U')\} \oplus F. \end{aligned}$$

Let $T = (-1 + (\bar{\pi})^{-1}) : \pi(U') \to X$. Since $\operatorname{im} T \subset \ker \pi$, we see that $\operatorname{im} T \subset V$. This gives us the desired decomposition of U'. One can now explicitly write a projection onto U'. Since $\pi(U') \subset U$ has finite codimension, it has a complement $C \subset U$ along with a projection $\pi_C : U \to C$ such that the complementary projection $(1 - \pi_C)$ has range equal to $\operatorname{im} \bar{\pi}$. A projection from X onto U' is now easily seen to be given by the map

$$(1 + T)(1 - \pi_C)\pi(1 - \pi_F) \oplus \pi_F. \tag{18.2}$$

This proves the lemma. □

Given a complemented subspace $U \subset X$, to compare other subspaces $U' \subset X$ with U, then we should not only study the projection of U' onto U but also onto a complement of U.

Definition 18.2 Let X be a Banach space. Two projections π and π' on X are *commensurate* if $\pi - \pi'$ is compact. Given a complemented subspace $U \subset X$, then a subspace U' is *commensurate* with U if its projection onto U is Fredholm and its projection onto some (hence any) complement of U is compact. We will also say that U' is a *compact perturbation* of U.

Corollary 18.3 *Let U and U' be as in Lemma 18.1. Then the subspace U' is commensurate with U if and only if the map T in (18.1) is compact. In this case, the space U is also commensurate with U'.*

Hence, being commensurate is a symmetric relation, and we may simply speak of two subspaces U and U' as being commensurate. The notion of commensurability obviously captures the notion of two subspaces being "close" to one another in a functional analytic sense. On the opposite spectrum, one may consider pairs of subspaces that form a direct sum decomposition modulo finite dimensional subspaces. More precisely, we have the following definition:

Definition 18.4 A pair of complemented subspaces (U, V) of a Banach space X is *Fredholm* if $U \cap V$ is finite dimensional and the algebraic sum $U + V$ is closed and has finite codimension. In this case, we say that (U, V) form a Fredholm pair, or more simply, that U and V are Fredholm (in X).

Together, the notion of a pair of subspaces being either commensurate or Fredholm will be very important in what we do. Next, we record the following technical lemmas concerning topological decompositions:

Lemma 18.5 *Let X and Y be Banach spaces, with $Y \subset X$ dense. Suppose $X = X_1 \oplus X_0$ and $Y \cap X_i \subseteq X_i$ is dense for $i = 0, 1$. Then if $Y \cap X_1$ and $Y \cap X_0$ are Fredholm in Y, then in fact $Y = (Y \cap X_1) \oplus (Y \cap X_0)$.*

Proof The hypotheses imply $Y = (Y \cap X_1) \oplus (Y \cap X_0) \oplus F$ where F is some finite dimensional subspace of Y. If we take the closure of this decomposition in X, we have $X \supseteq X_1 \oplus X_0 \oplus F$, which means $F = 0$. □

Lemma 18.6 *Let $X = X_1 \oplus X_0$ be a topological decomposition of X and let $\pi_i : X \to X_i$ be the coordinate projections. Let $V = V_1 \oplus V_0$, where $V_1 = V \cap X_1$ and $\pi_0 : V_0 \to X_0$ is Fredholm. If U is commensurate with V, then we can write $U = U_1 \oplus U_0$, where $U_1 = U \cap X_1$, $\pi_0 : U_0 \to X_0$ is Fredholm, and U_i is commensurate with V_i, $i = 0, 1$.*

Proof By the preceding analysis, since U is commensurate with V, there exist finite dimensional subspaces $F_1 \subset X$ and $F_2 \subset V$ and a compact operator $T : V/F_2 \to X$ such that $U = \{x + Tx : x \in V/F_2\} \oplus F_1$. For notational simplicity, let us suppose $F_1 = F_2 = 0$, since the conclusion is unaffected by finite dimensional errors. So then

$$\begin{aligned} U &= \{x + Tx : x \in V\} \\ &= \{x + Tx : x \in V_1\} + \{x + Tx : x \in V_0\} \\ &=: U_1' + U_0'. \end{aligned}$$

Since T is compact, then U_0' is commensurate with V_0 and since $\pi_0 : V_0 \to X_0$ is Fredholm, so is $\pi_0 : U_0' \to X_0$. Thus the map

$$\pi_0' = \pi_0 : U_0'/\ker \pi_0 \to \pi_0(U_0')$$

is an isomorphism. Let $V_1' \subset V_1$ be the subspace of finite codimension defined by

$$V_1' := \{x \in V_1 : \pi_0(Tx) \in \pi_0(U_0')\},$$

In other words, V_1' is the subspace of V_1 such that the space $\{x + Tx : x \in V_1'\} \subseteq U_1'$ differs from an element of X_1 by an element of U_0'. Indeed, we have

$$\{x + Tx - {\pi_0'}^{-1}\pi_0 T(x) : x \in V_1'\} \subseteq X_1$$

since it is annihilated by π_0. We thus have

$$U_1 = U \cap X_1 = \{x + Tx - {\pi_0'}^{-1}\pi_0 T(x) : x \in V_1'\} + \ker(\pi_0|_{U_0'}).$$

From this expression for U_1, it follows that U_1 is commensurate with V_1. Letting $U_0 = U_0' + \{x + Tx : x \in V_1/V_1'\}$, then $U = U_1 \oplus U_0$ and all the properties are satisfied. □

Lemma 18.7 *Let U_0, U_1, V_0, and V_1 be subspaces of a Banach space X such that we have the topological decompositions*

$$X = U_0 \oplus U_1 = V_0 \oplus V_1 \tag{18.3}$$

$$= V_0 \oplus U_1 = U_0 \oplus V_1. \tag{18.4}$$

Let π_{U_0,U_1} denote the projection onto U_0 through U_1 and similarly for other pairs of complementary spaces in the above. Since $\pi_{U_0,U_1} : V_0 \to U_0$ and $\pi_{U_1,U_0} : V_1 \to U_1$ are isomor-

phisms, then V_i *is the graph of a map* $T_{V_i} : U_i \to U_{i+1}$, $i = 0, 1 \bmod 2$.

(i) We have the following formulas:

$$\pi_{V_0,U_1} = (1 + T_{V_0})\pi_{U_0,U_1} \tag{18.5}$$
$$\pi_{U_1,V_0} = 1 - \pi_{V_0,U_1} \tag{18.6}$$
$$\pi_{V_0,V_1} = (1 + T_{V_0})\pi_{U_0,U_1}(1 + T_{V_1}\pi_{U_1,V_0})^{-1} \tag{18.7}$$

and likewise with the 0 *and* 1 *indices switched.*

(ii) If V_i *is commensurate with* U_i*, for* $i = 0, 1$*, then* π_{V_0,V_1} *is commensurate with* π_{U_1,U_0}*. This remains true even if we drop the assumption (18.4).*

Proof (i) Formula (18.5) is just the definition of T_{V_0}; indeed, this is the special case of the formula (18.1) when the map $\pi : U' \to U$ is an isomorphism. Formula (18.6) is tautological since U_1 and V_0 are complementary. It remains to establish (18.7). Let $\Lambda : X \to X$ be the isomorphism of X which maps V_0 identically to V_0 and U_1 to V_1 using the graph map T_{V_1}. In other words, Λ is given by

$$\begin{aligned}\Lambda &= \pi_{V_0,U_1} + (1 + T_{V_1})\pi_{U_1,V_0} \\ &= 1 + T_{V_1}\pi_{U_1,V_0}.\end{aligned}$$

The map π_{V_0,V_1} is now easily seen to be given by $\pi_{V_0,U_1}\Lambda^{-1}$, which yields (18.7). By symmetry, these formulas hold with 0 and 1 indices reversed.

(ii) In this case, the maps T_{V_i} are compact, $i = 0, 1$. It follows from (18.7) that $\pi_{V_0,V_1} - \pi_{U_0,U_1}$ is compact. If (18.4) does not hold, we proceed as follows. Let F denote the finite dimensional space spanned by the kernel and cokernel of the Fredholm maps $\pi_{U_0,U_1} : V_0 \to U_0$ and $\pi_{U_1,U_0} : V_1 \to U_1$. Let $\bar{X} = X/F$ be regarded as a subspace of X and let $\bar{\pi} : X \to \bar{X}$ be the projection through F. It follows that we can choose finite codimensional subspaces $U_i' \subseteq U_i$ and $V_i' \subseteq V_i$ such that, letting $\bar{U}_i = \pi_i(U_i')$ and $\bar{V}_i = \pi_i(V_i')$, we have

$$\bar{X} = \bar{U}_0 \oplus \bar{U}_1 = \bar{V}_0 \oplus \bar{V}_1.$$

By construction of $\bar{X}$, we also have

$$\bar{X} = \bar{V}_0 \oplus \bar{U}_1 = \bar{U}_0 \oplus \bar{V}_1,$$

since now $\bar{V}_i$ is a graph over $\bar{U}_i$. On $\bar{X}$, we can therefore conclude that the projections $\pi_{\bar{V}_1,\bar{V}_0}$ and $\pi_{\bar{U}_1,\bar{U}_0}$ are commensurate. These operators also act on X since we can define them to be zero on F, in which case, π_{V_0,V_1} and π_{U_0,U_1} are finite rank perturbations of $\pi_{\bar{V}_0,\bar{V}_1}$ and $\pi_{\bar{U}_0,\bar{U}_1}$, respectively. It now follows that π_{V_1,V_0} and π_{U_1,U_0} are also commensurate. □

Remark 18.8 In all applications, our Banach space X under consideration will be a function space of configurations on a manifold, and the compact operators that arise will be maps that smooth by a certain number of derivatives $\sigma \geq 0$ (e.g. the operator maps a Besov space $B^{s,p}$ to a more regular Besov space $B^{s+\sigma,p}$). In this way, if additionally we have that all finite dimensional subspaces which arise in the above analysis are spanned by elements that are smoother than those of X by σ derivatives, one can ensure that all compact perturbations occurring in the projections constructed in the above lemmas continue to be operators that are smoothing of order σ. In other words, the amount of smoothing is preserved in all our constructions.

The notion of commensurability of two spaces is one qualitative way of measuring two spaces as being close. Alternatively, we may regard two subspaces V_1 and V_2 of X as being close if V_2 is the graph over V_1 of a map with small norm, i.e. $V_2 = \{x + Tx : x \in V_1\}$ where $V_1^\perp$ is any fixed complement of V_1 and $T : V_1 \to V_1^\perp$ is an operator with small norm. If the norm of T is small enough, we can replace $V_1^\perp$ with X. This motivates the following definition:

Definition 18.9 (i) A *continuous family of subspaces* $\{V(\sigma)\}_{\sigma\in\mathfrak{X}}$ of X, where $\mathfrak{X}$ is a topological space, is a collection of complemented subspaces $V(\sigma)$ of X such that the following local triviality condition holds: for any $\sigma_0 \in \mathfrak{X}$, there exists an open neighborhood $U \ni \sigma_0$ in $\mathfrak{X}$ such that for all $\sigma \in U$, there exists a map $T_{\sigma_0}(\sigma) : V \to X$ such that the induced map

$$\begin{aligned} V(\sigma_0) &\to V(\sigma) \\ x &\mapsto x + T(\sigma)x \end{aligned} \tag{18.8}$$

is an isomorphism. The map $T_{\sigma_0}(\sigma)$ varies continuously in the operator norm topology with respect to $\sigma \in U$.

(ii) A *smooth family of subspaces* $V(t)$ of X, $t \in \mathbb{R}$, is a continuous family of subspaces for which $\mathfrak{X} = \mathbb{R}$ and the maps $T(t)$ in (18.8) vary smoothly in operator norm topology.

This definition is such that one can construct operators associated to a continuously varying family subspaces in a continuous way, e.g., projections onto such subspaces. Likewise for the smooth situation. To illustrate this, we state the following trivial lemma for small time intervals:

Lemma 18.10 *Let $V(t)$ be a continuous (smooth) family of subspaces of X, for $t \in \mathbb{R}$. Then for any $t_0 \in \mathbb{R}$, we can find an open interval I containing t_0, and a continuous (smooth) family of isomorphisms $\Phi(t) : X \to X$, $t \in I$, such that $\Phi(t)(V(t_0)) = V(t)$ for all $t \in \mathbb{R}$, with $\Phi(0) = \mathrm{id}$.*

Proof Without loss of generality, let $t_0 = 0$ and suppose we are in the smooth case, with the continuous case being the same. Let $V(0)^\perp$ be any complement of $V(0)$ in X. Then for small enough t, $V(t)$ is also a complement of $V(0)^\perp$, and we can define

$$\begin{aligned} \Phi(t) : V(0) \oplus V(0)^\perp &\to V(t) \oplus V(0)^\perp \\ (x, y) &\mapsto (x + T(t)x, y), \end{aligned}$$

where $x \mapsto x + T(t)x$ is the isomorphism from $V(0)$ to $V(t)$ given by the definition of V being a smooth family of subspaces of X. The maps $\Phi(t)$ are smooth since the $V(t)$ are. □

In other words, the family of spaces $V(t)$ has local trivializations given by the $\Phi(t)$ which identify the $V(t)$ with $V(t_0)$, for $t \in I$. Given a family of spaces complementary to the $V(t)$ and which vary smoothly, one can construct the $\Phi(t)$ for all t, but the above local result will suffice for our purposes.

19 Symplectic Linear Algebra

Let X be a real Banach space endowed with a skew-symmetric bilinear form ω. Then X is a *(weakly) symplectic Banach space* if ω is nondegenerate, i.e., the map $\omega : X \to X^*$ which assigns to $x \in X$ the linear functional $\omega(x, \cdot)$ is injective. If X is a Hilbert space and there exists an automorphism $J : X \to X$ such that $J^2 = -\mathrm{id}$ and $\omega(\cdot, J\cdot)$ is the inner product on X, we say that X is a *strongly symplectic Hilbert space* and that J is the compatible complex structure. (As a word of caution, many other authors define a symplectic Banach space to be one for which $\omega : X \to X^*$ is an isomorphism, but that will never be the case for us unless X is a strongly symplectic Hilbert space.)

Given any subspace V of a symplectic Banach space X, let $\mathrm{Ann}(V) \subset X$ denote its annihilator with respect to the symplectic form. A (co)isotropic subspace V is one for which $V \subseteq (\supseteq)$ $\mathrm{Ann}(V)$. A *Lagrangian subspace* L is an isotropic subspace which has an isotropic complement. This implies L is also coisotropic by the nondegeneracy of ω. In case X is a strongly symplectic Hilbert space, then in fact, an isotropic subspace is Lagrangian if and only if it is coisotropic, see [57]. In this latter case, any Lagrangian subspace L has an orthogonal Lagrangian complement JL.

The following procedure, known as symplectic reduction, is well-known in the context of Hilbert spaces (see e.g. [20, Proposition 6.12]):

Theorem 19.1 *(Symplectic Reduction) Let (X, ω) be a strongly symplectic Hilbert space with compatible complex structure J. Let $U \subseteq X$ be a closed coisotropic subspace. Let $L \subset X$ be a Lagrangian subspace such that $L + \mathrm{Ann}(U)$ is closed. Then $U \cap JU$ is a strongly symplectic Hilbert space and the orthogonal projection $\pi_{U\cap JU}$ onto $U \cap JU$, yields a map*

$$\pi_{U\cap JU} : L \cap U \to U \cap JU \tag{19.1}$$

whose image $\pi_{U\cap JU}(L \cap U)$ is a Lagrangian subspace of $U \cap JU$.

We call the map (19.1) the symplectic reduction of L with respect to U. For symplectic reduction on Banach spaces, we can generalize the above result as follows:

Corollary 19.2 *Let Y be a Banach space with $Y \subseteq X$ dense. Given any subspace $V \subset X$, define $V_Y := Y \cap V$. Suppose $\pi_{U\cap JU}$ and J map Y into itself and that L_Y and U_Y are dense in L and U, respectively. Suppose $\pi_{U\cap JU}(L_Y)$ and $J\pi_{U\cap JU}(L_Y)$ are Fredholm in $U_Y \cap JU_Y$. Then $\pi_{U\cap JU}(L_Y)$ and $J\pi_{U\cap JU}(L_Y)$ are complementary Lagrangian subspaces of the symplectic Banach space $U_Y \cap JU_Y$.*

Proof This follows from the previous theorem and Lemma 18.5. □

20 Banach Manifolds and The Inverse Function Theorem

Taking the usual definition of a finite dimensional manifold, one may replace all occurrences of Euclidean space with some other fixed Banach space, thereby obtaining the notion of a (smooth) Banach manifold. In other words, a Banach manifold, modeled on a Banach space X, is a Hausdorff topological space that is locally homeomorphic to X and whose transition maps are all diffeomorphisms[1].

In a similar way, one also obtains the notion of a (smooth) Banach submanifold of a Banach space. More precisely, we have the following definition:

Definition 20.1 Let X be a Banach space. A *Banach submanifold* M of X is a subspace of X (as a topological space) that satisfies the following. There exists a closed Banach subspace $Z \subset X$ such that at every point $u \in M$, there exists an open set V in X containing u and a diffeomorphism Φ from V onto an open neighborhood of 0 in X such that $\Phi(V \cap M) = \Phi(V) \cap Z$. We say that M is modeled on the Banach space Z.

We almost always drop explicit reference to the model Banach space Z, since it will be clear what this space is in practice. Of course, one can consider abstract Banach manifolds that do not come with a global embedding into a Banach space, but such a situation will not occur for us. The above definition coincides with the usual definition of a submanifold when X is a Euclidean space.

In the general situation above, we have no information about the local chart maps Φ. However, if M is defined in some natural way, say as the zero set of some function, one can construct a more concrete local model for M. The tools we use for this are the inverse and implicit function theorems in the general setting of Banach spaces. Below, we record these theorems, mostly to fix notation in applications. Let $X = X_0 \oplus X_1$ be a direct sum of Banach spaces and $f : X \to Y$ a smooth map of Banach spaces. For any $x \in X$, let $D_x f : X \to Y$ denote the Fréchet derivative of f at x.

Theorem 20.2 *Suppose $D_0 f : X \to Y$ is surjective, with $D_0 f : X_1 \to Y$ an isomorphism and $X_0 = \ker D_0 f$.*

(i) (Implicit Function Theorem) Choose V to be an open neighborhood of 0 in X such that $D_x f : X_1 \to Y$ remains an isomorphism for all $x \in V$. Then $M := f^{-1}(0) \cap V$ is a Banach submanifold of X modeled on X_0.

(ii) (Inverse Function Theorem) Define the smooth map $F : X_0 \oplus X_1 \to X$ by

$$F(x_0, x_1) = (x_0, (D_0 f|_{X_1})^{-1} f(x_0, x_1)).$$

Then $F(0) = 0$, $D_0 F = \mathrm{id}$, and shrinking V if necessary, we can arrange that both F and F^{-1} are diffeomorphisms onto their images when restricted to V. In this case, we have $M \subseteq F^{-1}(F(V) \cap X_0)$.

Definition 20.3 Let $M \subset X$ be a Banach submanifold and let $u \in M$ be any element, which without loss of generality, we let be 0. Given a function $f : X \to Y$ as in (i) above, we

[1] A map of Banach spaces is smooth if it is infinitely Fréchet differentiable. A diffeomorphism is a smooth map that has a smooth inverse.

say that f is a *local defining function* for M near u if there exists a neighborhood V of $u \in X$ such that $M_u := M \cap V$ is a Banach submanifold of X and satisfies $M_u = f^{-1}(0) \cap V$. In this case, the function F associated to f in Theorem 20.2(ii) is said to be a *local straightening map* for M at u. If we wish to emphasize our choice of V, we will say that F is a local straightening map *within the neighborhood* V.

The names we give for f and F are natural given their role in describing M. Namely, the manifold M_u, which is an open neighborhood of u in M, is the subset of V that lies in the preimage under f of the regular value $0 \in Y$. On the other hand, the map F is a local diffeomorphism of X which straightens out M_u to an open neighborhood $U := F(M_u)$ inside the tangent space $X_0 = T_uM$. Consequently, $F^{-1} : U \to M$ is a diffeomorphism of U onto its image M_u, an open neighborhood of $0 \in M$.

Definition 20.4 With the above notation, we call $F^{-1} : U \to M$ the *induced chart map* of the local straightening map F.

Thus, while a Banach submanifold has no distinguished choice of local charts near any given point, a local straightening map gives us a canonical choice for one. We will be consistently using this choice when constructing local chart maps for the Banach submanifolds we study.

21 Analyticity

Let X and Y be Banach spaces.

Definition 21.1 A function $F : X \to Y$ is said to be *analytic* at $x_0 \in X$ if there exists a neighborhood U of $x_0 \in X$ and symmetric multilinear maps $L_n : X^n \to Y$, $n \geq 0$, such that we have a power series expansion

$$F(x) = \sum_{n=0}^{\infty} L_n((x - x_0)^n), \tag{21.1}$$

where the series converges absolutely and uniformly for all $x \in U$. A function F is analytic on an open set U if it is analytic at every point of U.

Theorem 21.2 *(Analytic Inverse Function Theorem) Suppose $F : X \to Y$ is a map such that $F(0) = 0$, $D_0F : X \to Y$ is an isomorphism, and F is analytic at 0. Then there exists a neighborhood U of $0 \in Y$ such that $F^{-1} : U \to X$ is analytic. In particular,*

$$F^{-1}(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(D_0^n F^{-1})(x^n), \tag{21.2}$$

where $D_0^n F^{-1}$ is the nth Fréchet derivative of F^{-1} at 0.

A proof of the above theorem can be found in [8].

A nice property about an analytic functions is that there is a *fixed* neighborhood upon which each of its Fréchet derivatives are (uniformly) Lipschitz. Of course, for a finite dimensional Banach space, this is trivial since then the Banach space is locally compact. On an infinite dimensional Banach space, however, a smooth function and all its derivatives are locally Lipschitz, but the balls on which one has a Lipschitz estimate on the k-jet may depend on the value of k. However, if we have a locally convergent power series, the uniform convergence properties of the power series allows us to obtain a fixed ball on every k-jet of the function is Lipschitz.

However, there are some subtleties concerning power series on infinite dimensional real Banach spaces, since among other issues, there are several distinct radii of convergence that one must consider (see e.g. [7],[33]). Indeed, given a power series (21.1), its radius of (uniform) convergence about x_0 is easily seen to be

$$\rho := \frac{1}{\limsup_{n\to\infty} \|\tilde{L}_n\|^{1/n}} \tag{21.3}$$

where

$$\begin{aligned} \tilde{L}_n : X &\to Y \\ x &\mapsto L_n(x^n) \end{aligned}$$

is the degree n polynomial associated to the multilinear map L_n and $\|\tilde{L}_n\| := \sup_{|x|=1} |L(x^n)|$ is its norm. (Here, we use $\|\cdot\|$ to denote the norm both on X and Y, since there is no confusion as to which space elements belong.) On the other hand, one may consider the full multilinear map norm of L_n, namely

$$\|L_n\| := \sup_{|x_1|,\ldots,|x_n|=1} |L_n(x_1,\ldots,x_n)|.$$

A simple application of the polarization identity (see [7]) implies that for any symmetric n-linear map L_n, we have

$$\|L_n\| \leq \frac{n^n}{n!}\|\tilde{L}_n\|.$$

A simple consequence of this is that

$$\frac{1}{\limsup_{n\to\infty} \|L_n\|^{1/n}} \geq \rho/e. \tag{21.4}$$

For any $r > 0$, let $B_r(x_0)$ denote the open ball of radius r centered at r. With the above considerations, we have the following:

Proposition 21.3 *Consider a power series $F(x) = \sum_{n=0}^\infty L_n(x^n)$ centered at 0 and let $\rho > 0$ be its radius of convergence. Let $0 < r < \rho/e$. Then for any $x_0 \in B_r(0)$, we have that $D^k_{x_0}F$ is Lipschitz on $B_r(0)$ for all $k \geq 0$.*

Proof Differentiating the power series for $F(x)$ term-by-term, we have that

$$D^k_{x_0}F = \sum_{n\geq k} n(n-1)\cdots(n-k+1)L_n(x_0^{n-k})$$

where each of the $L_n(x_0^{n-k})$ are to be regarded as symmetric k-linear maps in the obvious way. This term-by-term differentiation is justified by the fact that

$$\|D^k_{x_0}F\| \le \sum_{n\ge k} n^k \|L_n\| \|x_0\|^{n-k} < \infty$$

via (21.4) and $|x_0| < r < \rho/e$. For all $y_1, y_0 \in X$, then from the usual formula $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1})$ for numbers $a, b \in \mathbb{R}$, we have

$$\begin{aligned} L_n(x_0^{n-k}, y_1^k) - L_n(x_0^{n-k}, y_0^k) &= L_n(x_0^{n-k}, (y_1 - y_0), y_1^{k-1}) + L_n(x_0^{n-k}, (y_1 - y_0), y_1^{k-2}, y_0) \\ &\quad + \cdots + L_n(x_0^{n-k}, (y_1 - y_0), y_0^{k-1}) \end{aligned}$$

since the L_n are symmetric. Thus, we have

$$|D^k_{x_0}F(y_1) - D^k_{x_0}F(y_0)| \le \left(\sum_{n\ge k} kn^k \|L_n\| |x_0|^{n-k} \max(|y_1|, |y_0|)^{k-1} \right) |y_1 - y_0|.$$

The above sum is uniformly bounded in terms of r for $r < \rho/e$, which shows that $D^k_{x_0}F$ is uniformly Lipschitz for $x_0 \in B_r(0)$ for every $k \ge 0$. □

22 Self-Adjointness

Let D denote any (unbounded) closed symmetric operator on a real[2] Hilbert space H with domain $\mathrm{Dom}(D) \subset H$. We wish to understand the self-adjoint extensions of D. Let D^* be the adjoint of D, and equip $\mathrm{Dom}(D^*)$ with the inner product

$$(x, y)_{D^*} := (x, y) + (D^*x, D^*y). \tag{22.1}$$

Then $\mathrm{Dom}(D^*)$ becomes a Hilbert space with this inner product and $\mathrm{Dom}(D)$ becomes a closed subspace since D is a closed operator.

There is standard way of describing all self-adjoint extensions of D. Namely, consider $\mathrm{Dom}(D^*)$ equipped with the skew-symmetric form

$$\omega(x, y) = (D^*x, y) - (x, D^*y). \tag{22.2}$$

Then $\mathrm{Dom}(D) \subset \mathrm{Dom}(D^*)$ is a closed isotropic space and $\mathrm{Dom}(D^*)/\mathrm{Dom}(D)$ is a (strongly) symplectic Hilbert space. Indeed, if we identify $\mathrm{Dom}(D^*)/\mathrm{Dom}(D)$ with the orthogonal complement $\mathrm{Dom}(D)^\perp$ in $\mathrm{Dom}(D^*)$, then

$$\mathrm{Dom}(D)^\perp = \{x : D^*x \in \mathrm{Dom}(D^*), D^*D^*x = -x\},$$

and we have the relation

$$\omega(x, D^*y) = (x, y)_{D^*}$$

[2]What follows can be generalized to complex Hilbert spaces, but for simplicity we will not do so here.

for all $x, y \in \mathrm{Dom}(D)^\perp$. Thus, D^* is a compatible complex structure for ω, and so in particular, $\mathrm{Dom}(D)^\perp$ is a symplectic Hilbert space. We now have the following fact:

Proposition 22.1 *The self-adjoint extensions of D are in one-to-one correspondence with Lagrangian subspaces of* $\mathrm{Dom}(D^*)/\mathrm{Dom}(D)$.

Let us specialize to the case where D is a Dirac operator acting on smooth sections $\Gamma(E)$ of a Clifford bundle E on a manifold X with boundary Σ (though much of what we will discuss applies to more general elliptic differential operators acting between sections of vector bundles). Let $D_{\min}$ be the Dirac operator D with domain $H_0^{1,2}(E) \subset H := L^2(E)$, the $H^{1,2}(X)$ closure of the compactly supported sections. Then $D_{\max}$ is a closed symmetric operator and the previous analysis apply. The adjoint operator $D_{\max} = D_{\min}^*$ has domain $\mathrm{Dom}(D_{\max}) \supseteq H^1(E)$ and we can understand the space $\mathrm{Dom}(D_{\max})/\mathrm{Dom}(D_{\min})$ as follows.[3]

If $x \in \mathrm{Dom}(D_{\max})$, then $D_{\max}x \in L^2(E)$ and so by the results of Section 15.3, we have a well-defined trace map $r(x) \in H^{-1/2,2}(E_\Sigma)$. Moreover, the kernel of this map is precisely $\mathrm{Dom}(D_{\min})$. Indeed, we have the elliptic estimate $\|x\|_{H^{1,2}(E)} \leq C(\|D_{\max}x\|_{L^2(E)} + \|x\|_{L^2(E)} + \|r(x)\|_{H^{1/2,2}(E)})$, and so elements of $\mathrm{Dom}(D_{\max})$ which lie in the kernel of r belong to $H^{1,2}(E_\Sigma)$ and have zero trace, and hence belong to $H_0^{1,2}(E)$. Thus, we have a continuous injection

$$r : \mathrm{Dom}(D_{\max})/\mathrm{Dom}(D_{\max}) \to H^{-1/2,2}(E_\Sigma).$$

The space $\mathrm{Dom}(D_{\max})/\mathrm{Dom}(D_{\max})$ is naturally a Hilbert space, as it is a quotient of the Hilbert space $\mathrm{Dom}(D_{\max})$ equipped with the graph inner product (22.1). Define the Hilbert space

$$H_{BV}(D) := r(\mathrm{Dom}(D_{\max})/\mathrm{Dom}(D_{\max})),$$

which we identify isometrically with $\mathrm{Dom}(D_{\max})/\mathrm{Dom}(D_{\max})$ via the map r, i.e.,

$$\|x\|^2_{H_{BV}(D)} = \inf_{y \in \mathrm{Dom}(D_{\max}) : r(y) = x} \left(\|y\|^2_{L^2(E)} + \|D_{\max}y\|^2_{L^2(E)} \right).$$

Thus, we have identified the space $\mathrm{Dom}(D_{\max})/\mathrm{Dom}(D_{\max})$ with the space $H_{BV}(D)$, which is contained in the boundary data space $H^{-1/2,2}(E_\Sigma)$. By the above proposition, it is the Lagrangian subspaces L of $H_{BV}(D)$ that yield for us self-adjoint extensions of $D_{\max}$. Namely, to L we assign the operator D_L with domain

$$\mathrm{Dom}(D_L) = \{x \in \mathrm{Dom}(D_{\max}) : r(x) \in L\}.$$

The space $H_{BV}(D)$ is convenient to work with since it lives entirely on the boundary and we can apply our understanding of boundary value problems for the Dirac operator to understand $H_{BV}(D)$.

We wish to understand $H_{BV}(D)$ more explicitly. Let $P^+ : \Gamma(E_\Sigma) \to \Gamma(E_\Sigma)$ be the Calderon projection of D, which we may take to be an orthogonal projection, and let $P^- = 1 - P^+$ be its complementary projection. Recall that P^+ is a projection onto the

[3] Observe that for both $D_{\max}$ and $D_{\min}$, these operators can both be interpreted as the operator D acting in the sense of distributions. Thus when we apply $D_{\max}$ or $D_{\min}$ to an element, there is no harm in just denoting the operator simply by D.

Cauchy data of elements of the kernel of D. Being a pseudodifferential operator, it extends to a bounded map on $H^s(E_\Sigma)$ for all $s \in \mathbb{R}$.

Lemma 22.2 *We have the topological decomposition*

$$H_{BV}(D) = H^{1/2,2}\mathrm{im}\, P^- \oplus H^{-1/2,2}\mathrm{im}\, P^+, \tag{22.3}$$

with each factor a Lagrangian subspace of the Hilbert space $H_{BV}(D)$.

Proof First note that $H^{1/2}(E_\Sigma) \subseteq H_{BV}(D)$, since $H^1(E) \subset \mathrm{Dom}(D_{\max})$. Given any $x \in H_{BV}(D) \subseteq H^{-1/2,2}(E)$, let $x^\pm = P^\pm x$. Since we have a Poisson operator P : $H^{-1/2,2}\mathrm{im}\, P^+ \to L^2(E)$ with range contained in the $\ker D_{\max} = L^2(\ker D)$, we have

$$\|x^+\|^2_{H_{BV}(D)} \leq \|Px^+\|^2_{L^2(E)} + \|DPx^+\|^2_{L^2(E)} = \|Px^+\|^2_{L^2(E)}.$$

Moreover, the map $P : H^{-1/2,2}\mathrm{im}\, P^+ \to L^2 \ker D$ is an isomorphism which inverts the map $r : L^2(\ker D) \to H^{-1/2,2}\mathrm{im}\, P^+$.[4] Thus, we see that $H^{-1/2,2}\mathrm{im}\, P^+$ is contained in $H_{BV}(D)$, and the $H_{BV}(D)$ topology on $H^{-1/2,2}\mathrm{im}\, P^+$ is equivalent to the $H^{-1/2,2}(E)$ topology. Finally, since P^+ defines an elliptic boundary condition for D and $P^+P^- = 0$, we have the elliptic estimate $\|y\|_{H^{1,2}(E)} \leq C(\|D_{\max} y\|_{L^2(E)} + \|y\|_{L^2(E)})$ for all y such that $r(y) \in P^-(H_{BV}(D))$. This shows that the $H_{BV}(D)$ topology on $P^-(H_{BV}(D))$ is the $H^{1/2,2}(E)$ topology. Writing a general element x as $x = x^+ + x^-$, the decomposition (22.3) now follows. For the final claim, we need to show that each factor in (22.3) is isotropic. That $H^{1/2,2}\mathrm{im}\, P^-$ and $H^{-1/2,2}\mathrm{im}\, P^+$ are complementary Lagrangian subspaces follows from Proposition 15.18, since the range of P^- is precisely JP^-. □

Corollary 22.3 *Let $\Pi^\pm$ be the projection onto the nonnegative and negative eigenspaces of the boundary tangential operator associated to D. Then*

$$H_{BV}(D) = H^{1/2,2}\mathrm{im}\, \Pi^- \oplus H^{-1/2,2}\mathrm{im}\, \Pi^+. \tag{22.4}$$

Proof This follows from the previous lemma and the fact that the projections $\Pi^\pm$ have the same principal symbol as $P^\pm$, respectively, which implies $\Pi^\pm - P^\pm$ is an operator of order -1. □

Though the corollary is not essential for our purposes, it has the aesthetic quality that the decomposition (22.4) is determined entirely by boundary data while the decomposition (22.3) depends on the nonlocal nature of the projections on $P^\pm$. Nevertheless, from these decompositions, we have a simple criterion for finding Lagrangians in $H_{BV}(D)$. Namely, we wish to investigate which Lagrangians in the subspace $H^{1/2,2}(E_\Sigma)$ (which inherits the symplectic form on $H_{BV}(D)$) yield Lagrangians in $H_{BV}(D)$.

Theorem 22.4 *Let $L \subset H^{1/2,2}(E_\Sigma)$ be a Lagrangian subspace. If L is Fredholm with $H^{1/2,2}\mathrm{im}\, P^+$ in $H^{1/2,2}(E_\Sigma)$, then $L \subset H_{BV}(D)$ is a Lagrangian subspace. Consequently, $D_L : \mathrm{Dom}(D_L) \subseteq H^{1,2}(E) \to L^2(E)$ is self-adjoint and Fredholm.*

[4]This is because Dirac operators satisfy unique continuation.

Proof From Proposition 15.18, we have that $H^{1/2,2}\mathrm{im}\,P^{\pm}$ are complementary Lagrangians inside $H^{1/2,2}(E_\Sigma)$. If L is Fredholm with $H^{1/2,2}\mathrm{im}\,P^+$, then by Lemma 18.1, this means we have

$$L = \mathrm{graph}(T) \oplus V^+$$

where $V^+ = L \cap H^{1/2,2}\mathrm{im}\,P^+$ is finite dimensional, and $T : H^{1/2,2}\mathrm{im}\,P^- \dashrightarrow H^{1/2,2}\mathrm{im}\,P^+$ is a bounded map whose domain has finite codimension in $H^{1/2,2}\mathrm{im}\,P^-$. Since L is a Lagrangian, then graph(T) is isotropic. Furthermore, the subspace V^+ is precisely the intersection of $H^{1/2,2}\mathrm{im}\,P^+$ with the (symplectic) annihilator of the domain of T. Indeed, if V^+ were smaller than this latter space, then L would be a proper subset of its annihilator, a contradiction. From this, we can form the topological decomposition

$$H^{1/2,2}(E) = L \oplus (V^- \oplus K) \tag{22.5}$$

where V^- is any finite dimensional complement of the domain of T in $H^{1/2,2}\mathrm{im}\,P^-$ (and thus $V^- \oplus V^+$ is symplectic) and K is any complement of V^+ in $H^{1/2,2}\mathrm{im}\,P^+$. In particular, we can choose K to be the intersection of $H^{1/2,2}\mathrm{im}\,P^+$ with the annihilator of V^-. With this choice of K, it follows that $V^- \oplus K$ is isotropic and hence a complementary Lagrangian to L in $H^{1/2,2}(E_\Sigma)$.

We take the closure of (22.5) in the $H_{BV}(D)$ topology. The space L remains the same, since V^+ is finite dimensional and graph(T) remains the same by Lemma 22.2. Since V^- is finite dimensional, the only space that changes is K, which becomes $H^{-1/2,2}K$ when we take closures. Thus, we have the decomposition

$$H_{BV}(D) = L \oplus (V^- \oplus H^{-1/2,2}K)$$

where L is isotropic and $V^- \oplus H^{-1/2,2}K$ is isotropic. This shows that in particular, L is a Lagrangian in $H_{BV}(D)$. The final statement now follows from Proposition 22.1 and Theorem 15.23. □

In other words, the theorem tells us that we can find Lagrangians in the large space $H_{BV}(D)$ by finding Lagrangians in the smaller space $H^{1/2,2}(E)$. We need this result in the proof of Theorem 11.7 in Part III.

Index of Notation

$\mathfrak{a}$	A vortex, p. 83.
A	A typical spin^c connection, usually on a 4-manifold, p. 110. Also an elliptic differential operator in Part IV.
$\mathcal{A}(X)$	The space of spin^c connections on the manifold X, p. 18.
B	A typical spin^c connection, usually on a 3-manifold, p. 19. Also a boundary condition in Part IV, p. 201.
$B^{s,p}$, $B^{(s_1,s_2),p}$	A Besov space and anisotropic Besov space, respectively. Used as a prefix, it denotes closure with respect to said topology, p. 178, 184.
$\mathfrak{B}(X)$	The quotient of the configuration space $\mathfrak{C}(X)$ by the gauge group $\mathcal{G}(X)$, p. 83.
C	A spin^c connection on a 2-manifold or an unspecified constant, p. 77.
$\mathcal{C}$	A Coulomb slice in $\mathcal{T}$, p. 31.
CSD^{Σ}	The Chern-Simons-Dirac functional on Σ, p. 77.
$\mathfrak{C}(X)$	The smooth configuration space of spin^c connections and spinors on the manifold X, p. 19.
$\mathfrak{C}^{s,p}(X)$	The $B^{s,p}(X)$ configuration space on X, p. 21.
$\mathfrak{C}^{s;\delta}(Y)$	The configuration space on a cylindrical end manifold Y with elements differing from a time-translation vortex by an element of $H^{s;\delta}(Y)$, p. 93.
$\mathbf{d}_{\bullet}$	The operator associated to the infinitesimal action of the gauge group at $\bullet \in \mathfrak{C}(X)$, p. 26.
$\mathbf{d}_{\bullet}^{*}$	The formal adjoint of $\mathbf{d}_{\bullet}$, p. 27.
$E_{\partial X}$	Abbreviation for the restriction of a bundle E on X to ∂X.
$E_{\bullet}$	A chart map for $\bullet \in \mathfrak{M}$, $\mathcal{M}$, or $\mathcal{L}$, p. 60, 70.
$E_{\bullet}^{1}$	The nonlinear part of the chart map $E_{\bullet}$, p. 60, 70.
$\mathcal{E}_{\gamma}$	Chart map for a path $\gamma \in \mathrm{Maps}(I, \mathcal{M})$, p. 134.
$\mathcal{E}(\gamma)$	The energy of a configuration γ in Part II, p. 86.
$F_{(B,\Psi)}$	A local straightening map for $\mathfrak{M}$ at (B, Ψ), p. 57.

$F_{\Sigma,(B,\Psi)}$	A local straightening map for $\mathfrak{L}$ at $r_\Sigma(B,\Psi)$, p. 68.
γ	In Part II, shorthand for a configuration (B,Ψ), p. 83. In Part IV, a path of configurations.
$\gamma_{\mathfrak{a}}$	In Part II, the constant path determined by the vortex $\mathfrak{a}$, p. 83.
$\check{\gamma}(t)$	In Part II, the path of configurations determined by γ, p. 83.
$\mathcal{G}(X)$	The gauge group of transformations on X, p. 22.
$\mathcal{G}_\partial(X)$	The gauge group of transformations that is the identity on ∂X, p. 24.
$\widehat{\ }$	Given an operator T, then $\widehat{T}$ is the induced slicewise operator, p. 124.
$H^{s,p}$, $H^{(s_1,s_2),p}$	A Bessel potential and anisotropic Bessel potential space, respectively (otherwise known as fractional Sobolev spaces.) Used as a prefix, it denotes closure with respect to said topology, p. 178, 184.
$H^{s;\delta}$	A weighted Sobolev space, p. 92.
H^s	Abbreviation for $H^{s,2}$.
$\mathcal{H}_{(B,\Psi)}$	The Hessian of a configuration $(B,\Psi) \in \mathfrak{C}(Y)$, p. 31.
$\widetilde{\mathcal{H}}_{(B,\Psi)}$	The augmented Hessian of a configuration (B,Ψ), p. 32.
$\overline{\mathcal{H}}_{(B,\Psi)}$	The extended Hessian of a configuration (B,Ψ), p. 96.
$\mathcal{H}_{2,(C_0,\Upsilon_0)}$	The Hessian of a configuration $(C_0,\Upsilon_0) \in \mathfrak{C}(\Sigma)$, p. 81.
$\mathcal{J}_{(B,\Psi)}$, $\mathcal{J}_{(B,\Psi),t}$	The subspace of $\mathcal{T}$ given by the infinitesimal action of $\mathcal{G}(Y)$ and $\mathcal{G}_\partial(Y)$, respectively, p. 26.
J_Σ	The compatible complex structure on $\mathcal{T}_\Sigma$, p. 37
$\tilde{J}_\Sigma$	The compatible complex structure on $\widetilde{\mathcal{T}}_\Sigma$, p. 37.
$\mathcal{K}_{(B,\Psi)}$, $\mathcal{K}_{(B,\Psi),n}$	The orthogonal complement of $\mathcal{J}_{(B,\Psi)}$ and $\mathcal{J}_{(B,\Psi),t}$ in $\mathcal{T}_{(B,\Psi)}$, respectively, p. 26.
$\mathcal{K}(Y)$	The bundle over $\mathfrak{C}(Y)$ whose fiber over (B,Ψ) is $\mathcal{K}_{(B,\Psi)}$, p. 29.
$\mathcal{L}(Y)$, $\mathcal{L}^{s-1/p,p}(Y)$	The tangential boundary values of the space of monopoles $\mathfrak{M}$ and $\mathfrak{M}^{s,p}$ on Y, respectively, p. 20.
$\mathrm{Maps}^{(s_1,s_2),p}(I,\mathfrak{X})$	The space of maps from I into a space $\mathfrak{X}$ in an anisotropic Besov space topology $B^{(s_1,s_2),p}$, p. 122.
$\mathfrak{M}(Y)$, $\mathfrak{M}^{s,p}(Y)$	When Y is compact, the space of all monopoles in $\mathfrak{C}(Y)$ and $\mathfrak{C}^{s,p}(Y)$, respectively, p. 20. When Y has cylindrical ends, finite energy is imposed, p. 92.
$\mathcal{M}(Y)$, $M^{s,p}(Y)$	When Y is compact, the space of all monopoles on Y in $\mathfrak{C}(Y)$ and $\mathfrak{C}^{s,p}(Y)$, respectively, that are in global Coulomb gauge, p. 20.
$M(Y)$	The moduli space of gauge-equivalence classes of all finite energy monopoles on Y, p. 93.

μ	The moment map on $\mathfrak{C}(\Sigma)$ in Part II, p. 77. The Chern-Simons-Dirac 1-form in Part IV, p. 115.
ν	The outward unit normal vector field to Y, p. 32.
ω	The symplectic form on $\mathcal{T}_\Sigma$, p. 37.
$\tilde{\omega}$	The symplectic form on $\widetilde{\mathcal{T}}_\Sigma$, 37.
$P^+_{(B,\Psi)}$	The "Calderon projection" of the Hessian $\mathcal{H}_{(B,\Psi)}$, p. 42.
$\widetilde{P}^+_{(B,\Psi)}$	The Calderon projection of the augmented Hessian $\widetilde{\mathcal{H}}_{(B,\Psi)}$, p. 41.
$P_{(B,\Psi)}$	The "Poisson operator" of the Hessian $\mathcal{H}_{(B,\Psi)}$, p. 42.
$\widetilde{P}_{(B,\Psi)}$	The Poisson operator of the augmented Hessian $\widetilde{\mathcal{H}}_{(B,\Psi)}$, p. 41.
$\tilde{\partial}_\infty$	The limiting value map, p. 92.
∂_∞	The limiting value map on gauge-equivalence classes, p. 94.
$\Pi_{\mathcal{K}_{(B,\Psi)}}$	The projection onto $\mathcal{K}_\bullet$ through $\mathcal{J}_{\bullet,t}$, p. 29.
Ψ	A spinor, usually on 3-manifold, 18.
Φ	A spinor, usually on a 4-manifold, 110.
ρ	Clifford multiplication on Y, 18.
q	The quadratic map associated to the Seiberg-Witten map SW_3, p. 56.
r_Σ	The tangential restriction map, p. 15, 31.
r	The full restriction map, p. 32.
#	Some pointwise bilinear multiplication operation.
$*$	The Hodge star operator on Y.
$\check{*}$	The Hodge star operator on $\Sigma = \partial Y$.
$\mathfrak{s}$	A spin^c structure, p. 18
$\mathcal{S}$	The spinor bundle on Y, p. 18
SW_k	The Seiberg-Witten vector field in k-dimensions, $k = 2, 3, 4$, p. 77, 19, 114.
$\mathcal{T}_\bullet$	The tangent space $T_\bullet\mathfrak{C}(X)$ for a configuration $\bullet \in \mathfrak{C}(X)$, p. 26.
$\mathcal{T}$, $\mathcal{T}^{s,p}$	The space $\Omega^1(Y; i\mathbb{R}) \oplus \Gamma(\mathcal{S})$ and its $B^{s,p}(Y)$ closure, isomorphic to any tangent space of $\mathfrak{C}(Y)$ and $\mathfrak{C}^{s,p}(Y)$, respectively, p. 30.
$\mathcal{T}_\Sigma$, $\mathcal{T}^{s,p}_\Sigma$	The space $\Omega^1(\Sigma; i\mathbb{R}) \oplus \Gamma(\mathcal{S}_\Sigma)$ and its $B^{s,p}(\Sigma)$ closure, isomorphic to any tangent space of $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}^{s,p}(\Sigma)$, respectively, p. 3.3.
$\widetilde{\mathcal{T}}$	The augmented space $\widetilde{\mathcal{T}} \oplus \Omega^0(Y; i\mathbb{R})$, p. 32.
$\widetilde{\mathcal{T}}_\Sigma$	The augmented space $\mathcal{T}_\Sigma \oplus \Omega^0(\Sigma; i\mathbb{R}) \oplus \Omega^0(\Sigma; i\mathbb{R})$, p. 32.
Υ	A spinor on a 2-manifold, p. 77.
$\mathscr{V}_k(\Sigma)$	The space of all k-vortices on Σ, p. 79.

$\mathcal{V}_k(\Sigma)$	The moduli space of k-vortices on Σ, p. 79.
Y	A 3-manifold.
X	A manifold or a Banach space.
$\mathcal{X}$	A Banach space.
$\tilde{X}^{s,p}_{(B,\Psi)}$, $X^{s,p}_{(B,\Psi)}$	Subspaces of $\widetilde{\mathcal{T}}^{s,p}$ and $\mathcal{T}^{s,p}$ on which $\widetilde{\mathcal{H}}_{(B,\Psi)}$ and $\mathcal{H}_{(B,\Psi)}$ are invertible, respectively, p. 49.

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